MATH 221 · FALL 2026

Midterm 1 Study Guide

ABOUT THE EXAM

Midterm 1 is on **Friday, September 25th** in class. You will have 50 minutes. No calculators, notes, phones, or other aids are allowed, so practice doing the arithmetic by hand. The exam has **7 problems** worth 100 points, plus an optional extra credit problem. Show your work on every problem. The exam covers everything we have done so far, which corresponds to Sections 1.1 through 2.4 of the textbook.

WHAT YOU NEED TO KNOW

1. **Many ways to say the same thing.** (1.1, 2.3) Translate freely between a system of linear equations, its augmented matrix, a vector equation $x_1v_1 + \cdots + x_nv_n = b$, a matrix equation $Ax = b$, and the question of whether b is in $\text{Span}\{v_1, \ldots, v_n\}$. Know what A, x, and b are in each case, and know how to go in every direction.

2. **Row reduction.** (1.2) The three row operations: swap two rows, scale a row by a nonzero number, add a multiple of one row to another. Row echelon form and reduced row echelon form, pivots, pivot columns, and free variables. Be able to row reduce a 3×5 augmented matrix by hand without errors.

3. **Counting solutions.** (1.2) A system has no solutions exactly when its echelon form has a row $[\, 0 \cdots 0 \mid c \,]$ with $c \neq 0$. A consistent system has exactly one solution when every variable is a pivot variable, and infinitely many when there is at least one free variable. You can read all of this off an echelon form, so you do not need to go all the way to reduced row echelon form to count solutions.

4. **The general solution.** (1.3) Starting from reduced row echelon form, set each free variable equal to a parameter and solve for the pivot variables to write down every solution at once.

5. **Vectors and their geometry.** (2.1) Vector addition (tip to tail), scalar multiplication (stretching, shrinking, reversing), subtraction, and linear combinations, all of which you should be able to draw.

6. **Span.** (2.2) The definition $\text{Span}\{v_1, \ldots, v_k\} := \{c_1v_1 + \cdots + c_kv_k : c_i \in \mathbb{R}\}$. To decide whether b is in the span, row reduce the augmented matrix $[\, v_1 \ \cdots \ v_k \mid b \,]$. The number of ways to write b as a linear combination of the v_i is the number of solutions of that system.

WHAT YOU NEED TO KNOW (CONTINUED)

7. **Homogeneous systems.** (2.4) A system $Ax = 0$ always has the solution $x = 0$. Its solution set is a span, so any linear combination of solutions is again a solution. Know how to find the solution set of $Ax = 0$ and write it as a span.

8. **Matrix multiplication.** (2.3) When AB is defined and what size it is. The entry of AB in row i and column j is row i of A times column j of B. The vector Ax is the linear combination of the columns of A with coefficients from x.

PRACTICE PROBLEMS

The problems below are similar in style to the problems on the exam. Work them by hand under exam conditions, then check your row reductions with the Row Reducer. The problem sets are also good practice.

PROBLEM 1

Consider the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 5 \\ 3x_1 + x_2 &= -1 \\ -x_1 + 4x_3 &= 2. \end{aligned}$$

Do not solve the system.

1. Write the augmented matrix of the system.
2. Write the system as a vector equation.
3. Write the system as a matrix equation $Ax = b$. Say explicitly what A, x, and b are.
4. Rewrite the question "Does this system have a solution?" as a question about whether one vector is in the span of some other vectors.
5. Now go the other way. Write the question "Is $(2, -1, 3)$ in $\text{Span}\{(1, 0, 2), (3, 1, -1)\}$?" as a system of linear equations, and write down its augmented matrix.

PROBLEM 2

Each of the following augmented matrices is in row echelon form. For each one, decide whether the corresponding system has no solutions, one solution, or infinitely many solutions. If the system is consistent, say which variables are free.

1. $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 6 \end{array}\right]$ For this one, also find the solution.

2. $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

3. $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

PROBLEM 3

Let

$$A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 4 & 2 & -2 \\ 0 & -3 & 1 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 4 \\ -2 & 0 \end{bmatrix}.$$

1. What size is AB? Is BA defined?
2. Compute the entry in the 2nd row and 2nd column of AB.
3. Compute the entry in the 3rd row and 1st column of AB.
4. Compute the whole matrix AB.

PROBLEM 4

Let

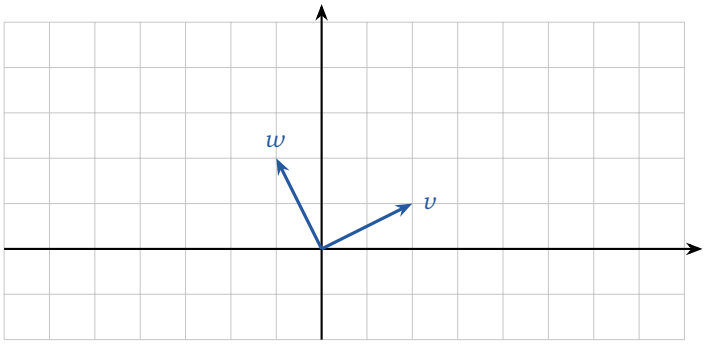
$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \qquad x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

1. Compute AB and BA. Are they equal?
2. Compute Ax, and then write Ax as a linear combination of the columns of A.
3. Find all vectors y with $Ay = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

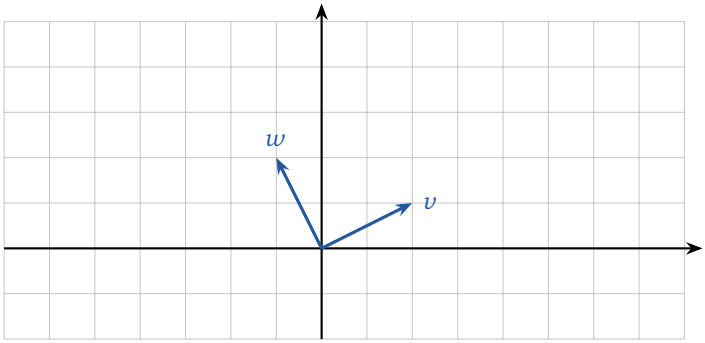
PROBLEM 5

The vectors v and w are drawn on each grid below. On each grid, draw the requested vector as an arrow starting at the origin.

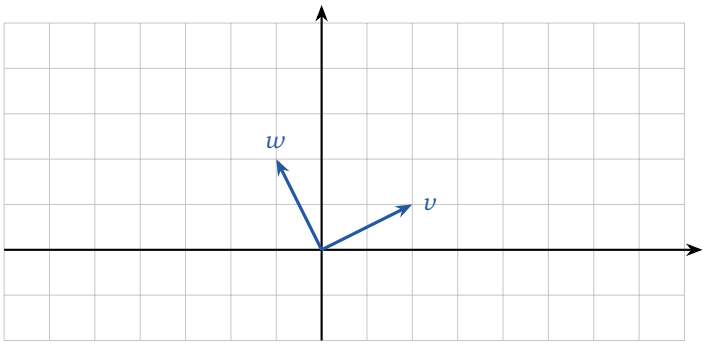
1. $v + w$



2. $2v - w$



3. $2w - v$



4. Describe Span$\{v, w\}$ geometrically.

PROBLEM 6

In each part, how many ways are there to write b as a linear combination of v_1, v_2, v_3: none, one, or infinitely many? Justify your answer. If b is in Span$\{v_1, v_2, v_3\}$, also write it explicitly as a linear combination.

1. $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$

2. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

PROBLEM 7

Find the general solution of the vector equation

$$x_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix}.$$

PROBLEM 8

1. Let
$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 3 & 1 \end{bmatrix}.$$
Find all solutions of the homogeneous equation $Ax = 0$, and write the solution set as a span.

2. Let B be a matrix for which the vectors
$$p = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \qquad q = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$
are both solutions of the homogeneous equation $Bx = 0$. Find a solution of $Bx = 0$ with $x_1 = 4$ and $x_3 = 4$.

3. Explain why the solution set of the equation $x_1 + x_2 = 1$ is not a span.

PROBLEM 9

Let
$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 4 \\ h \end{bmatrix},$$
where h is an unknown number.

1. For which values of h is b in $\text{Span}\{v_1, v_2\}$?

2. For that value of h, write b as a linear combination of v_1 and v_2.