MATH 221 · FALL 2026

Midterm 2 Study Guide

ABOUT THE EXAM

Midterm 2 is on **Friday, November 13th** in class. You will have 50 minutes. No calculators, notes, phones, or other aids are allowed, so practice doing the arithmetic by hand. The exam has **7 problems** worth 100 points, plus an optional extra credit problem. Show your work on every problem. The exam covers everything since Midterm 1: Problem Sets 4 through 8 and the determinant material from class (row and column operations and cofactor expansion). This corresponds to Sections 2.4 through 4.3 of the textbook.

WHAT YOU NEED TO KNOW

1. **Solution sets.** (2.4) If v_0 is one solution of $Ax = b$, then every solution has the form $x = v_0 + \text{Nul}(A)$. Write the general solution in parametric vector form $v_0 + t_1 v_1 + \cdots + t_k v_k$, with one parameter for each free variable. The solution set of $Ax = b$ is a subspace only when $b = 0$.

2. **Linear independence.** (2.5) The vectors $v_1, \ldots, v_k$ are linearly independent when $x_1 v_1 + \cdots + x_k v_k = 0$ if and only if $x_1 = \cdots = x_k = 0$. To decide, row reduce the matrix with columns $v_1, \ldots, v_k$: they are independent exactly when every column is a pivot column. A nonzero solution of the homogeneous system gives a linear dependence relation. More than n vectors in $\mathbb{R}^n$ are always dependent. If a collection is dependent, then *some* vector in the list is a linear combination of the others.

3. **Null space and column space.** (2.6) $\text{Nul}(A)$ is the solution set of $Ax = 0$, and $\text{Col}(A)$ is the span of the columns of A. A basis for $\text{Nul}(A)$ comes from the parametric vector form of the solutions of $Ax = 0$. A basis for $\text{Col}(A)$ is the pivot columns of A itself (not of its row reduction). To check whether v is in $\text{Nul}(A)$, compute Av. To check whether b is in $\text{Col}(A)$, decide whether $Ax = b$ is consistent.

4. **Basis and dimension.** (2.7) A basis of a subspace V is a linearly independent collection of vectors that spans V, and $\dim V$ is the number of vectors in any basis. The dimension of $\text{Span}\{v_1, \ldots, v_k\}$ is the number of pivots in the row reduction of the matrix with the v_i as columns; the pivot columns form a smaller collection with the same span. A collection of n vectors in $\mathbb{R}^n$ form a basis when the matrix with those columns has n pivots, which happens exactly when its determinant is nonzero.

5. **Coordinates.** (2.8) If $\mathcal{B} = \{v_1, \ldots, v_k\}$ is a basis of V and $x = c_1 v_1 + \cdots + c_k v_k$, then $[x]_{\mathcal{B}}$ is the column vector of the c_i. Find $[x]_{\mathcal{B}}$ by solving this system of equations, and go back from $[x]_{\mathcal{B}}$ to x by computing the linear combination.

WHAT YOU NEED TO KNOW (CONTINUED)

6. **The Rank-Nullity Theorem.** (2.9) $\operatorname{rank}(A) = \dim \operatorname{Col}(A)$ is the number of pivots, and $\operatorname{nullity}(A) = \dim \operatorname{Nul}(A)$ is the number of free variables. For an $m \times n$ matrix, $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$. Use this to deduce facts about A from its size and partial information.

7. **Linear transformations and their matrices.** (3.1, 3.3) $T_A(x) = Ax$, and the columns of A are $T_A(e_1), \ldots, T_A(e_n)$. If $T_A(u_1) = w_1$ and $T_A(u_2) = w_2$ for a basis u_1, u_2 in $\mathbb{R}^2$, then $AU = W$, so $A = WU^{-1}$, where U and W have these vectors as columns. A linear transformation is determined by what it does to a basis: to find $T_A(u)$, write u in $\mathcal{B}$-coordinates and use linearity. This is how we computed reflections and projections in a plane in $\mathbb{R}^3$.

8. **Injective, surjective, and bijective.** (3.2) T_A is injective exactly when every column of A has a pivot (equivalently $\operatorname{Nul}(A) = \{0\}$), and surjective exactly when every row has a pivot (equivalently $\operatorname{Col}(A) = \mathbb{R}^m$). A matrix with more columns than rows is never injective, and one with more rows than columns is never surjective.

9. **Inverses and the Invertible Matrix Theorem.** (3.5, 3.6) Find A^{-1} by row reducing $[\, A \mid I_n \,]$ to $[\, I_n \mid A^{-1} \,]$, and know the 2×2 inverse formula. If A is invertible, then $Ax = b$ has the unique solution $x = A^{-1}b$, and $(AB)^{-1} = B^{-1}A^{-1}$. For a square matrix A, all of the following are the same: A is invertible, T_A is bijective, T_A is injective, T_A is surjective, A has n pivots, the columns of A are linearly independent, the columns of A span $\mathbb{R}^n$, and $\det(A) \neq 0$.

10. **Determinants.** (4.1) The determinant of an upper triangular matrix is the product of its diagonal entries, and $\det(AB) = \det(A)\det(B)$. Swapping two rows changes the sign, multiplying a row by c multiplies the determinant by c, and adding a multiple of one row to another does not change it. Compute determinants by row reducing to upper triangular form while keeping track of these changes. Consequences: $\det(A^{-1}) = 1/\det(A)$, $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix, and $\det(A) = 0$ exactly when A is not invertible (for example, when $Ax = 0$ for some nonzero x).

11. **Column operations.** Column operations change the determinant in exactly the same way as the corresponding row operations. Use whichever is more convenient, and you may mix them.

12. **Cofactor expansion.** (4.2) Expand along any row or column, with signs in a checkerboard pattern starting with + in the top left corner. Choose a row or column with many zeros. Cofactor expansion is especially useful for small matrices and for matrices with an unknown entry.

13. **Determinants and area.** (4.3) T_A multiplies areas by $|\det(A)|$. $\det(A) < 0$ means T_A reverses orientation, so a picture comes out as its mirror image, and $\det(A) = 0$ means T_A squishes the plane onto a line or a point.

PRACTICE PROBLEMS

The problems below are meant to provide practice for the midterm. I suggest you work them by hand under exam conditions, then check your row reductions with the Row Reducer. If you finish all of these, I suggest reviewing your problem sets. Note that the solutions for problem sets are posted on the course website.

PROBLEM 1

Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 4 & -1 & 1 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}.$$

1. Find a basis for $\text{Nul}(A)$.
2. Find a basis for $\text{Col}(A)$, and find $\text{rank}(A)$ and $\text{nullity}(A)$.
3. Is T_A injective? Is it surjective? Explain.
4. Is $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in $\text{Col}(A)$?
5. Find a vector in $\mathbb{R}^3$ that is not in $\text{Col}(A)$.

PROBLEM 2

Let A be the matrix from Problem 1, and let $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

1. Find the general solution of $Ax = b$ in parametric vector form (that is, in the form $v_0 + t_1 v_1 + \cdots + t_k v_k$).
2. Is the solution set of $Ax = b$ a subspace of $\mathbb{R}^5$? Explain.

PROBLEM 3

For each collection of vectors, decide whether it is linearly independent. If it is not, find a linear dependence relation among the vectors ($c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ with c_1, c_2, c_3 not all zero).

1. $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$

2. $v_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$

PROBLEM 4

1. Is $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ a basis of $\mathbb{R}^3$?
2. Is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ a basis of $\mathbb{R}^3$?
3. The vectors $v_1 = (1, 1, 0, 0)$ and $v_2 = (0, 1, 1, 0)$ are linearly independent. Find a basis of $\mathbb{R}^4$ that contains v_1 and v_2.

PROBLEM 5

Let $V = \text{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}, \qquad v_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}.$$

The vectors v_1, v_2 are not multiples of each other, so $\mathcal{B} = \{v_1, v_2\}$ is a basis of V.

1. The vector $w = \begin{bmatrix}2\\3\\1\end{bmatrix}$ is in V. Find $[w]_{\mathcal{B}}$.
2. Is $u = \begin{bmatrix}1\\2\\3\end{bmatrix}$ in V?
3. Find the vector x with $[x]_{\mathcal{B}} = \begin{bmatrix}-1\\4\end{bmatrix}$.

PROBLEM 6

1. A is a 5×7 matrix with $\text{rank}(A) = 4$. What is $\text{nullity}(A)$?
2. A is a 4×4 matrix, and the solution set of $Ax = 0$ is a line through the origin. What is $\text{rank}(A)$?
3. A is a 3×5 matrix, and $Ax = b$ is consistent for every b in $\mathbb{R}^3$. Describe the solution set of $Ax = 0$.

4. A is a 4×7 matrix. Explain why $\text{Nul}(A)$ cannot be a plane through the origin.

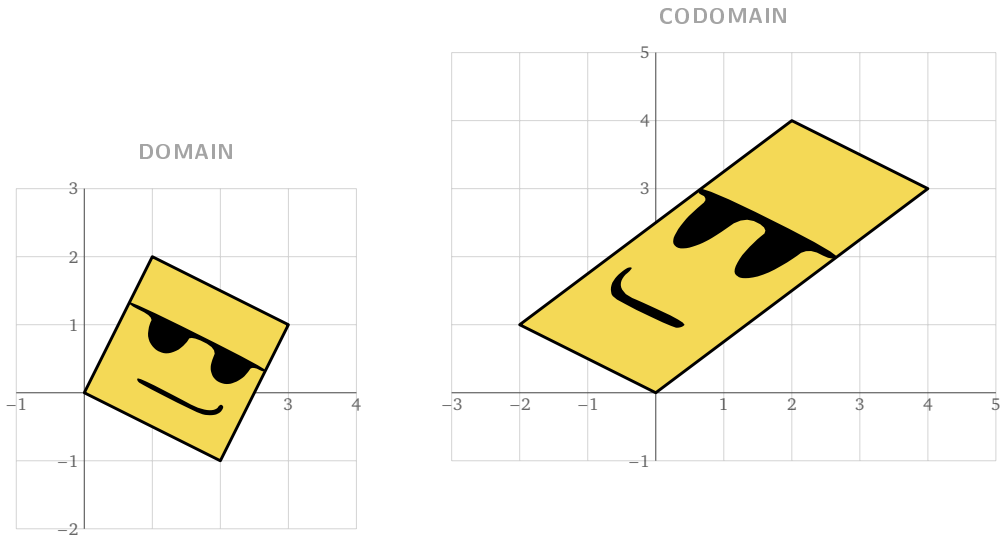
PROBLEM 7

In each part, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix A.

1. $T_A(e_1) = (2, -1)$ and $T_A(e_2) = (0, 3)$.
2. T_A rotates the plane counterclockwise by $90°$.
3. T_A first rotates the plane counterclockwise by $90°$ and then reflects it across the line $y = x$.
4. $T_A(1, 2) = (3, 0)$ and $T_A(1, 1) = (1, 1)$.

PROBLEM 8

The face on the left sits on the square with corners at the origin, $u_1 = (2, -1)$, $u_2 = (1, 2)$, and $u_1 + u_2 = (3, 1)$. The matrix A is such that T_A maps the face on the left to the face on the right.



1. Find A.
2. Find $\det(A)$. How can you see the sign of $\det(A)$ in the picture?
3. The face on the left has area 5. What is the area of the face on the right?

PROBLEM 9

For each matrix A below, decide whether T_A is injective, surjective, and/or bijective. Justify your answers.

1. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

PROBLEM 10

1. Find A^{-1}, where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$.

2. Find the inverse of $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

3. For which values of k is $\begin{bmatrix} k & 2 \\ 3 & k+1 \end{bmatrix}$ invertible?

4. Suppose A and B are 2×2 matrices with

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Find $(AB)^{-1}$.

PROBLEM 11

Compute each determinant.

1. $\det \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix}$, using row operations

2. $\det \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \\ 0 & 4 & 2 \end{bmatrix}$, using cofactor expansion

3. $\det \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix}$, using cofactor expansion

PROBLEM 12

Suppose that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5.$$

Find the determinant of each matrix below.

1. $\begin{bmatrix} c & b+4c & 2a \\ f & e+4f & 2d \\ i & h+4i & 2g \end{bmatrix}$
2. $\begin{bmatrix} 2a & b & c-a \\ 2d & e & f-d \\ 2g & h & i-g \end{bmatrix}$
3. $\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$

PROBLEM 13

1. Find all values of c for which the equation $Ax = b$ has exactly one solution for every b in $\mathbb{R}^3$, where
$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}.$$
2. Find all values of h for which the vectors
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}, \quad \begin{bmatrix} h \\ 1 \\ 0 \end{bmatrix}$$
are linearly dependent.

PROBLEM 14

Let A and B be 4×4 matrices with $\det(A) = -2$ and $\det(B) = 3$. Find each of the following.

1. $\det(AB^2)$
2. $\det(A^{-1}B)$
3. $\det(3A)$
4. $\det(B^{-1}AB)$

PROBLEM 15

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example for which it fails.

1. If A is a 3×3 matrix and T_A is injective, then T_A is surjective.
2. If v_1, v_2, v_3 are linearly dependent, then v_3 is in $\text{Span}\{v_1, v_2\}$.
3. If A and B are 2×2 matrices with $AB = 0$, then $A = 0$ or $B = 0$.
4. If the columns of a 4×4 matrix A span $\mathbb{R}^4$, then $Ax = b$ has exactly one solution for every b in $\mathbb{R}^4$.

PROBLEM 16

Let P be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

lie in P, and the vector $w = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is perpendicular to P. Then $\mathcal{B} = \{v_1, v_2, w\}$ is a basis of $\mathbb{R}^3$. Let

$$u = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

1. Find $[u]_{\mathcal{B}}$.
2. Let A be the matrix such that T_A projects $\mathbb{R}^3$ onto P, so $T_A(v) = v$ for every v in P and $T_A(w) = 0$. Find $T_A(u)$.
3. Find $\det(A)$, where A is the matrix from part (b).
4. Let R be the matrix such that T_R reflects $\mathbb{R}^3$ across P, so $T_R(v) = v$ for every v in P and $T_R(w) = -w$. Find $T_R(u)$.
5. Find $\det(R)$, where R is the matrix from part (d).