MATH 221 · FALL 2026

Midterm 2 Study Guide

SOLUTIONS

ABOUT THE EXAM

Midterm 2 is on **Friday, November 13th** in class. You will have 50 minutes. No calculators, notes, phones, or other aids are allowed, so practice doing the arithmetic by hand. The exam has **7 problems** worth 100 points, plus an optional extra credit problem. Show your work on every problem. The exam covers everything since Midterm 1: Problem Sets 4 through 8 and the determinant material from class (row and column operations and cofactor expansion). This corresponds to Sections 2.4 through 4.3 of the textbook.

WHAT YOU NEED TO KNOW

1. **Solution sets.** (2.4) If v_0 is one solution of $Ax = b$, then every solution has the form $x = v_0 + \mathrm{Nul}(A)$. Write the general solution in parametric vector form $v_0 + t_1 v_1 + \cdots + t_k v_k$, with one parameter for each free variable. The solution set of $Ax = b$ is a subspace only when $b = 0$.

2. **Linear independence.** (2.5) The vectors $v_1, \ldots, v_k$ are linearly independent when $x_1 v_1 + \cdots + x_k v_k = 0$ if and only if $x_1 = \cdots = x_k = 0$. To decide, row reduce the matrix with columns $v_1, \ldots, v_k$: they are independent exactly when every column is a pivot column. A nonzero solution of the homogeneous system gives a linear dependence relation. More than n vectors in $\mathbb{R}^n$ are always dependent. If a collection is dependent, then *some* vector in the list is a linear combination of the others.

3. **Null space and column space.** (2.6) $\mathrm{Nul}(A)$ is the solution set of $Ax = 0$, and $\mathrm{Col}(A)$ is the span of the columns of A. A basis for $\mathrm{Nul}(A)$ comes from the parametric vector form of the solutions of $Ax = 0$. A basis for $\mathrm{Col}(A)$ is the pivot columns of A itself (not of its row reduction). To check whether v is in $\mathrm{Nul}(A)$, compute Av. To check whether b is in $\mathrm{Col}(A)$, decide whether $Ax = b$ is consistent.

4. **Basis and dimension.** (2.7) A basis of a subspace V is a linearly independent collection of vectors that spans V, and $\dim V$ is the number of vectors in any basis. The dimension of $\mathrm{Span}\{v_1, \ldots, v_k\}$ is the number of pivots in the row reduction of the matrix with the v_i as columns; the pivot columns form a smaller collection with the same span. A collection of n vectors in $\mathbb{R}^n$ form a basis when the matrix with those columns has n pivots, which happens exactly when its determinant is nonzero.

5. **Coordinates.** (2.8) If $\mathcal{B} = \{v_1, \ldots, v_k\}$ is a basis of V and $x = c_1 v_1 + \cdots + c_k v_k$, then $[x]_{\mathcal{B}}$

WHAT YOU NEED TO KNOW (CONTINUED)

is the column vector of the c_i. Find $[x]_{\mathcal{B}}$ by solving this system of equations, and go back from $[x]_{\mathcal{B}}$ to x by computing the linear combination.

6. **The Rank-Nullity Theorem.** (2.9) $\operatorname{rank}(A) = \dim \operatorname{Col}(A)$ is the number of pivots, and $\operatorname{nullity}(A) = \dim \operatorname{Nul}(A)$ is the number of free variables. For an $m \times n$ matrix, $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$. Use this to deduce facts about A from its size and partial information.

7. **Linear transformations and their matrices.** (3.1, 3.3) $T_A(x) = Ax$, and the columns of A are $T_A(e_1), \ldots, T_A(e_n)$. If $T_A(u_1) = w_1$ and $T_A(u_2) = w_2$ for a basis u_1, u_2 in $\mathbb{R}^2$, then $AU = W$, so $A = WU^{-1}$, where U and W have these vectors as columns. A linear transformation is determined by what it does to a basis: to find $T_A(u)$, write u in $\mathcal{B}$-coordinates and use linearity. This is how we computed reflections and projections in a plane in $\mathbb{R}^3$.

8. **Injective, surjective, and bijective.** (3.2) T_A is injective exactly when every column of A has a pivot (equivalently $\operatorname{Nul}(A) = \{0\}$), and surjective exactly when every row has a pivot (equivalently $\operatorname{Col}(A) = \mathbb{R}^m$). A matrix with more columns than rows is never injective, and one with more rows than columns is never surjective.

9. **Inverses and the Invertible Matrix Theorem.** (3.5, 3.6) Find A^{-1} by row reducing $[\, A \mid I_n \,]$ to $[\, I_n \mid A^{-1} \,]$, and know the 2×2 inverse formula. If A is invertible, then $Ax = b$ has the unique solution $x = A^{-1}b$, and $(AB)^{-1} = B^{-1}A^{-1}$. For a square matrix A, all of the following are the same: A is invertible, T_A is bijective, T_A is injective, T_A is surjective, A has n pivots, the columns of A are linearly independent, the columns of A span $\mathbb{R}^n$, and $\det(A) \neq 0$.

10. **Determinants.** (4.1) The determinant of an upper triangular matrix is the product of its diagonal entries, and $\det(AB) = \det(A)\det(B)$. Swapping two rows changes the sign, multiplying a row by c multiplies the determinant by c, and adding a multiple of one row to another does not change it. Compute determinants by row reducing to upper triangular form while keeping track of these changes. Consequences: $\det(A^{-1}) = 1/\det(A)$, $\det(cA) = c^n \det(A)$ for an $n \times n$ matrix, and $\det(A) = 0$ exactly when A is not invertible (for example, when $Ax = 0$ for some nonzero x).

11. **Column operations.** Column operations change the determinant in exactly the same way as the corresponding row operations. Use whichever is more convenient, and you may mix them.

12. **Cofactor expansion.** (4.2) Expand along any row or column, with signs in a checkerboard pattern starting with + in the top left corner. Choose a row or column with many zeros. Cofactor expansion is especially useful for small matrices and for matrices with an unknown entry.

13. **Determinants and area.** (4.3) T_A multiplies areas by $|\det(A)|$. $\det(A) < 0$ means T_A reverses orientation, so a picture comes out as its mirror image, and $\det(A) = 0$ means T_A squishes the plane onto a line or a point.

PRACTICE PROBLEMS

The problems below are meant to provide practice for the midterm. I suggest you work them by hand under exam conditions, then check your row reductions with the Row Reducer. If you finish all of these, I suggest reviewing your problem sets. Note that the solutions for problem sets are posted on the course website.

PROBLEM 1

Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 4 & -1 & 1 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}.$$

1. Find a basis for $\text{Nul}(A)$.

SOLUTION

We row reduce A.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 4 & -1 & 1 & 4 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots are in columns 1 and 3, so $x_2 = r$, $x_4 = s$, and $x_5 = t$ are free. The nonzero rows say $x_1 = -2r - s - t$ and $x_3 = -s + 2t$, so the solutions of $Ax = 0$ are

$$r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

These three vectors form a basis for $\text{Nul}(A)$. □

2. Find a basis for $\text{Col}(A)$, and find $\text{rank}(A)$ and $\text{nullity}(A)$.

SOLUTION

The pivot columns of A are columns 1 and 3, so a basis for $\text{Col}(A)$ is

$$\begin{bmatrix}1\\2\\1\end{bmatrix}, \qquad \begin{bmatrix}-1\\-1\\0\end{bmatrix}.$$

So $\text{rank}(A) = 2$ and $\text{nullity}(A) = 3$ (and $2 + 3 = 5$, the number of columns). □

3. Is T_A injective? Is it surjective? Explain.

SOLUTION

T_A is not injective, since only 2 of the 5 columns have pivots (so $\text{Nul}(A) \neq \{0\}$). T_A is not surjective, since only 2 of the 3 rows have pivots (so $\text{Col}(A)$ is a plane, not all of $\mathbb{R}^3$). □

4. Is $b = \begin{bmatrix}1\\3\\2\end{bmatrix}$ in $\text{Col}(A)$?

SOLUTION

Yes. We look for x_1, x_2 with $x_1(1,2,1) + x_2(-1,-1,0) = (1,3,2)$. The third coordinate gives $x_1 = 2$, the first then gives $x_2 = 1$, and the second coordinate checks: $4 - 1 = 3$. So b is the first column of A times 2 plus the third column of A. □

5. Find a vector in $\mathbb{R}^3$ that is not in $\text{Col}(A)$.

SOLUTION

We find exactly which b are in $\text{Col}(A)$ by row reducing $[\, a_1 \; a_3 \mid b \,]$, where a_1, a_3 are the pivot columns:

$$\left[\begin{array}{cc|c}1 & -1 & b_1\\2 & -1 & b_2\\1 & 0 & b_3\end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c}1 & -1 & b_1\\0 & 1 & b_2 - 2b_1\\0 & 1 & b_3 - b_1\end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c}1 & -1 & b_1\\0 & 1 & b_2 - 2b_1\\0 & 0 & b_1 - b_2 + b_3\end{array}\right].$$

So b is in $\text{Col}(A)$ exactly when $b_1 - b_2 + b_3 = 0$. For example, $(1,0,0)$ is not in $\text{Col}(A)$. □

PROBLEM 2

Let A be the matrix from Problem 1, and let $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

1. Find the general solution of $Ax = b$ in parametric vector form (that is, in the form $v_0 + t_1 v_1 + \cdots + t_k v_k$).

SOLUTION

We row reduce $[\,A \mid b\,]$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 3 & 1 \\ 2 & 4 & -1 & 1 & 4 & 3 \\ 1 & 2 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

With $x_2 = r$, $x_4 = s$, $x_5 = t$ free, the nonzero rows say $x_1 = 2 - 2r - s - t$ and $x_3 = 1 - s + 2t$, so

$$x = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

In other words, $x = v_0 + \mathrm{Nul}(A)$ with $v_0 = (2, 0, 1, 0, 0)$, using the basis of $\mathrm{Nul}(A)$ from Problem 1. □

2. Is the solution set of $Ax = b$ a subspace of $\mathbb{R}^5$? Explain.

SOLUTION

No. Every subspace contains 0, but $x = 0$ is not a solution, since $A0 = 0 \neq b$. □

PROBLEM 3

For each collection of vectors, decide whether it is linearly independent. If it is not, find a linear dependence relation among the vectors ($c_1v_1 + c_2v_2 + c_3v_3 = 0$ with c_1, c_2, c_3 not all zero).

1. $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\3\\6\end{bmatrix}$

SOLUTION

We row reduce the matrix with columns v_1, v_2, v_3.

$$\begin{bmatrix}1&2&0\\2&1&3\\3&0&6\end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix}1&2&0\\0&-3&3\\3&0&6\end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix}1&2&0\\0&-3&3\\0&-6&6\end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix}1&2&0\\0&-3&3\\0&0&0\end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix}1&2&0\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}$$

Column 3 has no pivot, so the vectors are **linearly dependent**. Setting the free variable $x_3 = 1$ gives $x_1 = -2$ and $x_2 = 1$, so

$$-2v_1 + v_2 + v_3 = 0.$$ □

2. $v_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$

SOLUTION

We row reduce the matrix with columns v_1, v_2, v_3 to echelon form.

SOLUTION (CONTINUED)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 2R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column has a pivot, so the vectors are **linearly independent**. □

PROBLEM 4

1. Is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ a basis of $\mathbb{R}^3$?

SOLUTION

Yes. Three vectors in $\mathbb{R}^3$ form a basis exactly when the matrix with these columns has nonzero determinant. Expanding along the first row,

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 1 \cdot (1 - 0) - 2 \cdot (0 - 1) + 0 = 3 \neq 0.$$ □

2. Is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ a basis of $\mathbb{R}^3$?

SOLUTION

No. We row reduce the matrix with these columns.

SOLUTION (CONTINUED)

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are only 2 pivots, so the vectors are linearly dependent (in fact $-2v_1 + v_2 + v_3 = 0$) and do not form a basis. □

3. The vectors $v_1 = (1, 1, 0, 0)$ and $v_2 = (0, 1, 1, 0)$ are linearly independent. Find a basis of $\mathbb{R}^4$ that contains v_1 and v_2.

SOLUTION

There are many answers. One is v_1, v_2, e_3, e_4. To check it, we row reduce the matrix with these columns:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are 4 pivots, so these four vectors are linearly independent and form a basis of $\mathbb{R}^4$. □

PROBLEM 5

Let $V = \text{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The vectors v_1, v_2 are not multiples of each other, so $\mathcal{B} = \{v_1, v_2\}$ is a basis of V.

1. The vector $w = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is in V. Find $[w]_{\mathcal{B}}$.

SOLUTION

We solve $x_1 v_1 + x_2 v_2 = w$.

SOLUTION (CONTINUED)

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

So $x_1 = 2$ and $x_2 = 3$, and

$$[w]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

□

2. Is $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in V?

SOLUTION

No. We try to solve $x_1 v_1 + x_2 v_2 = u$.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array}\right]$$

The last row says $0 = 3$, so the system is inconsistent and u is not in V. □

3. Find the vector x with $[x]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

SOLUTION

$$x = -v_1 + 4v_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -6 \end{bmatrix}.$$

□

PROBLEM 6

1. A is a 5×7 matrix with $\text{rank}(A) = 4$. What is $\text{nullity}(A)$?

SOLUTION

By the Rank-Nullity Theorem, $\text{nullity}(A) = 7 - 4 = 3$. □

2. A is a 4×4 matrix, and the solution set of $Ax = 0$ is a line through the origin. What is $\text{rank}(A)$?

SOLUTION

nullity$(A) = 1$, so rank$(A) = 4 - 1 = 3$. (In particular, A is not invertible.) □

3. A is a 3×5 matrix, and $Ax = b$ is consistent for every b in $\mathbb{R}^3$. Describe the solution set of $Ax = 0$.

SOLUTION

Col$(A) = \mathbb{R}^3$, so rank$(A) = 3$ and nullity$(A) = 5 - 3 = 2$. The solution set of $Ax = 0$ is a plane through the origin in $\mathbb{R}^5$. □

4. A is a 4×7 matrix. Explain why Nul(A) cannot be a plane through the origin.

SOLUTION

A has 4 rows, so it has at most 4 pivots and rank$(A) \leq 4$. So nullity$(A) = 7 - \text{rank}(A) \geq 3$, but a plane is 2-dimensional. □

PROBLEM 7

In each part, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Find the matrix A.

1. $T_A(e_1) = (2, -1)$ and $T_A(e_2) = (0, 3)$.

SOLUTION

The columns of A are $T_A(e_1)$ and $T_A(e_2)$:

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}.$$ □

2. T_A rotates the plane counterclockwise by 90°.

SOLUTION

The rotation sends e_1 to $e_2 = (0, 1)$ and e_2 to $-e_1 = (-1, 0)$, so

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ □

3. T_A first rotates the plane counterclockwise by 90° and then reflects it across the line $y = x$.

SOLUTION

The reflection across $y = x$ swaps the coordinates, so its matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Doing the rotation first and the reflection second gives the product with the reflection on the left:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So T_A is the reflection across the x-axis. □

4. $T_A(1, 2) = (3, 0)$ and $T_A(1, 1) = (1, 1)$.

SOLUTION

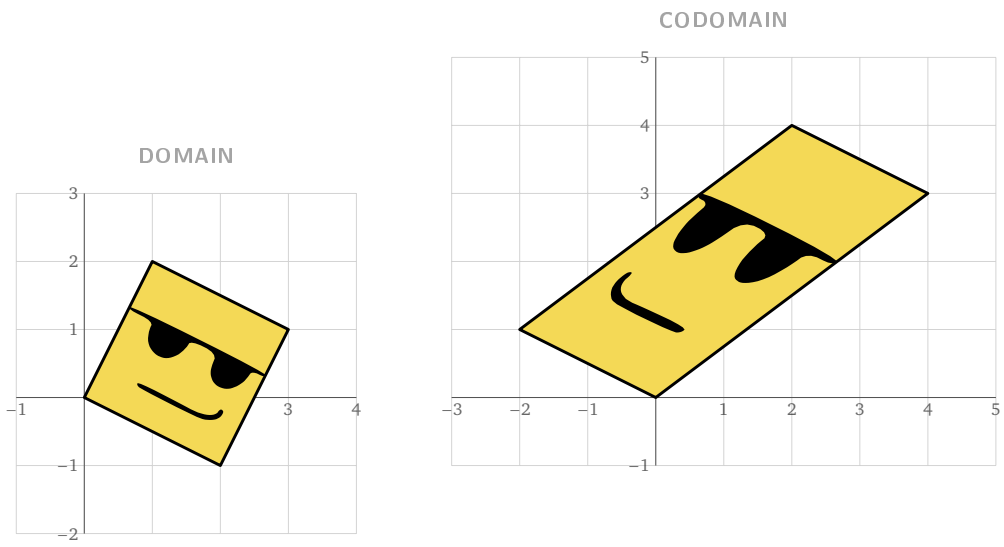
Let $U = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $W = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AU = W$, so $A = WU^{-1}$. By the 2×2 inverse formula,

$$U^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Check: $A(1, 2) = (3, 0)$ and $A(1, 1) = (1, 1)$. □

PROBLEM 8

The face on the left sits on the square with corners at the origin, $u_1 = (2, -1)$, $u_2 = (1, 2)$, and $u_1 + u_2 = (3, 1)$. The matrix A is such that T_A maps the face on the left to the face on the right.



1. Find A.

SOLUTION

Reading off the picture, $T_A(u_1) = w_1 = (-2, 1)$ and $T_A(u_2) = w_2 = (4, 3)$. With $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$ we have $AU = W$, so $A = WU^{-1}$:

$$U^{-1} = \frac{1}{4+1}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \qquad A = \frac{1}{5}\begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & 10 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}. \qquad \square$$

2. Find $\det(A)$. How can you see the sign of $\det(A)$ in the picture?

SOLUTION

$\det(A) = (0)(1) - (2)(1) = -2$. The determinant is negative because T_A reverses orientation: the face on the right is a mirror image of the face on the left. (Going counterclockwise around the left square we meet u_1 before u_2, but going counterclockwise around the right parallelogram we meet w_2 before w_1.) $\square$

3. The face on the left has area 5. What is the area of the face on the right?

SOLUTION

T_A multiplies areas by $|\det(A)| = 2$, so the face on the right has area 10. $\square$

PROBLEM 9

For each matrix A below, decide whether T_A is injective, surjective, and/or bijective. Justify your answers.

1. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

SOLUTION

A is already in echelon form, with pivots in both rows, so T_A is **surjective**. Column 3 has no pivot, so T_A is **not injective** (and not bijective). $\square$

2. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

SOLUTION

Row reducing,

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Both columns have pivots, so T_A is **injective**. The third row has no pivot, so T_A is **not surjective** (and not bijective). □

3. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

SOLUTION

Expanding along the first row,

$$\det(A) = 2(4-1) - 1(2-0) + 0 = 4 \neq 0.$$

So A is invertible and T_A is **bijective** (both injective and surjective). □

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

SOLUTION

The third column is the sum of the first two, so the columns are linearly dependent and T_A is **not injective**. Since A is square, the Invertible Matrix Theorem says T_A is also **not surjective**. □

PROBLEM 10

1. Find A^{-1}, where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}$.

SOLUTION

We row reduce $[\, A \mid I_3 \,]$.

SOLUTION (CONTINUED)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} -3 & 2 & -2 \\ 0 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}.$$

Check: $AA^{-1} = I_3$. □

2. Find the inverse of $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

SOLUTION

By the 2×2 inverse formula,

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{6-5} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}.$$ □

3. For which values of k is $\begin{bmatrix} k & 2 \\ 3 & k+1 \end{bmatrix}$ invertible?

SOLUTION

The determinant is $k(k+1) - 6 = k^2 + k - 6 = (k+3)(k-2)$. The matrix is invertible exactly when this is nonzero, that is, when $k \neq 2$ and $k \neq -3$. □

4. Suppose A and B are 2×2 matrices with

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Find $(AB)^{-1}$.

SOLUTION

The inverse of a product is the product of the inverses in the opposite order:

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}. \qquad \square$$

PROBLEM 11

Compute each determinant.

1. $\det \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix}$, using row operations

SOLUTION

We row reduce to upper triangular form.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Row replacements do not change the determinant, and there was one swap, so

$$\det = -\big(1 \cdot 1 \cdot 1 \cdot (-3)\big) = 3. \qquad \square$$

2. $\det \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \\ 0 & 4 & 2 \end{bmatrix}$, using cofactor expansion

SOLUTION

Expanding along the first row, which has a zero,

$$\det = 2 \det \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} - 0 + 1 \det \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} = 2(2+4) + (12-0) = 24. \qquad \square$$

3. $\det \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix}$, using cofactor expansion

SOLUTION

Row 3 has only one nonzero entry, the 2 in column 2, whose sign in the checkerboard pattern is $(-1)^{3+2} = -1$. Expanding along row 3 and then along the first row of the 3×3 matrix,

$$\det = -2 \det \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 1 \\ 4 & 0 & 3 \end{bmatrix} = -2\big(1(3-0) - 2(9-4) + 0\big) = -2(-7) = 14. \qquad \square$$

PROBLEM 12

Suppose that

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5.$$

Find the determinant of each matrix below.

1. $\begin{bmatrix} c & b+4c & 2a \\ f & e+4f & 2d \\ i & h+4i & 2g \end{bmatrix}$

SOLUTION

We use column operations, starting from the matrix with determinant 5. Swapping columns 1 and 3 changes the sign, giving -5. Adding 4 times the new column 1 to column 2 does not change the determinant. Multiplying column 3 by 2 doubles it. So the determinant is -10. $\square$

2. $\begin{bmatrix} 2a & b & c-a \\ 2d & e & f-d \\ 2g & h & i-g \end{bmatrix}$

SOLUTION

Subtracting column 1 from column 3 does not change the determinant, and then multiplying column 1 by 2 doubles it. So the determinant is 10. □

3. $\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$

SOLUTION

Swapping rows 1 and 2, and then swapping rows 2 and 3, turns the original matrix into this one. Each swap changes the sign, so the determinant is $(-1)^2 \cdot 5 = 5$. □

PROBLEM 13

1. Find all values of c for which the equation $Ax = b$ has exactly one solution for every b in $\mathbb{R}^3$, where

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}.$$

SOLUTION

$Ax = b$ has exactly one solution for every b exactly when A is invertible, that is, when $\det(A) \neq 0$. Adding columns 2 and 3 to column 1 does not change the determinant, and makes every entry of column 1 equal to $c + 2$. Factoring $c + 2$ out of column 1 and then subtracting row 1 from rows 2 and 3,

$$\det(A) = (c+2)\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} = (c+2)\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & c-1 & 0 \\ 0 & 0 & c-1 \end{bmatrix} = (c+2)(c-1)^2.$$

So $Ax = b$ has exactly one solution for every b exactly when $c \neq 1$ and $c \neq -2$. □

2. Find all values of h for which the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ h \\ 1 \end{bmatrix}, \quad \begin{bmatrix} h \\ 1 \\ 0 \end{bmatrix}$$

are linearly dependent.

SOLUTION

The vectors are linearly dependent exactly when the matrix with these columns has determinant 0. Expanding along the first row,

$$\det\begin{bmatrix}1 & 2 & h\\ 0 & h & 1\\ 1 & 1 & 0\end{bmatrix} = 1(0-1) - 2(0-1) + h(0-h) = 1 - h^2.$$

So the vectors are linearly dependent exactly when $h = 1$ or $h = -1$. □

PROBLEM 14

Let A and B be 4×4 matrices with $\det(A) = -2$ and $\det(B) = 3$. Find each of the following.

1. $\det(AB^2)$

SOLUTION

$\det(AB^2) = \det(A)\det(B)^2 = -2 \cdot 9 = -18.$ □

2. $\det(A^{-1}B)$

SOLUTION

$$\det(A^{-1}B) = \frac{\det(B)}{\det(A)} = -\frac{3}{2}.$$ □

3. $\det(3A)$

SOLUTION

$3A$ is A with each of its four rows multiplied by 3, so $\det(3A) = 3^4 \det(A) = 81 \cdot (-2) = -162$. □

4. $\det(B^{-1}AB)$

SOLUTION

$$\det(B^{-1}AB) = \frac{1}{\det(B)}\det(A)\det(B) = \det(A) = -2.$$ □

PROBLEM 15

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example for which it fails.

1. If A is a 3×3 matrix and T_A is injective, then T_A is surjective.

   SOLUTION

   True. If T_A is injective, every column has a pivot, so A has 3 pivots. Since A has 3 rows, every row has a pivot too, so T_A is surjective. □

2. If v_1, v_2, v_3 are linearly dependent, then v_3 is in $\text{Span}\{v_1, v_2\}$.

   SOLUTION

   False. Take $v_1 = e_1$, $v_2 = 2e_1$, and $v_3 = e_2$ in $\mathbb{R}^2$. They are dependent, since $2v_1 - v_2 = 0$, but $\text{Span}\{v_1, v_2\}$ is the x-axis, which does not contain v_3. □

3. If A and B are 2×2 matrices with $AB = 0$, then $A = 0$ or $B = 0$.

   SOLUTION

   False. Take

   $$A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but neither matrix is zero. □

4. If the columns of a 4×4 matrix A span $\mathbb{R}^4$, then $Ax = b$ has exactly one solution for every b in $\mathbb{R}^4$.

   SOLUTION

   True. If the columns span $\mathbb{R}^4$, then A has a pivot in every row, so it has 4 pivots. Since A is square, it then has a pivot in every column as well, so A is invertible and $x = A^{-1}b$ is the only solution. □

PROBLEM 16

Let P be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

lie in P, and the vector $w = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is perpendicular to P. Then $\mathcal{B} = \{v_1, v_2, w\}$ is a basis of $\mathbb{R}^3$. Let

$$u = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

1. Find $[u]_{\mathcal{B}}$.

SOLUTION

We row reduce $[\, v_1 \; v_2 \; w \mid u \,]$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 2 & -1 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & -2 & -4 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & -2 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -6 & -6 \end{array}\right] \xrightarrow{R_3 \to -\frac{1}{6} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So $u = 2v_1 - v_2 + w$, and

$$[u]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}. \qquad \square$$

2. Let A be the matrix such that T_A projects $\mathbb{R}^3$ onto P, so $T_A(v) = v$ for every v in P and $T_A(w) = 0$. Find $T_A(u)$.

SOLUTION

$$T_A(u) = 2T_A(v_1) - T_A(v_2) + T_A(w) = 2v_1 - v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}. \qquad \square$$

3. Find $\det(A)$, where A is the matrix from part (b).

SOLUTION

$Aw = 0$ with $w \neq 0$, so A is not invertible and $\det(A) = 0$. □

4. Let R be the matrix such that T_R reflects $\mathbb{R}^3$ across P, so $T_R(v) = v$ for every v in P and $T_R(w) = -w$. Find $T_R(u)$.

SOLUTION

$$T_R(u) = 2v_1 - v_2 - w = \begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}0\\2\\1\end{bmatrix} - \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\-3\end{bmatrix}.$$ □

5. Find $\det(R)$, where R is the matrix from part (d).

SOLUTION

Let $M = \begin{bmatrix}v_1 & v_2 & w\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$. The columns of RM are $Rv_1 = v_1$, $Rv_2 = v_2$, and $Rw = -w$, which are the columns of MD, so $RM = MD$. Taking determinants, $\det(R)\det(M) = \det(M)\det(D)$. Since $\mathcal{B}$ is a basis, M is invertible and $\det(M) \neq 0$, so $\det(R) = \det(D) = -1$. □