MATH 221 · FALL 2026

Problem Set 1: Warm-Up

DUE Friday, September 4th at 11:59pm

The world was so recent that many things lacked names, and in order to indicate them it was necessary to point.

From *One Hundred Years of Solitude* by Gabriel García Márquez

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

In this first problem set we try to get the juices flowing when it comes to thinking about systems of linear equations. There will be some computation, but it is more about thinking geometrically and conceptually.

PROBLEM 1.1

For each of the following scenarios, provide an example or explain why it is impossible.

1. A system of three linear equations in three variables which has infinitely many solutions, but none of the equations is a scaled copy of another equation (for example, $2x + 4y = 6$ is a scaled copy of $x + 2y = 3$).
2. A system of linear equations which has a solution, but after removing an equation from the system, it does not have any solutions.
3. Four planes in $\mathbb{R}^3$ such that any three of the planes intersect in a common point, but the four planes have no common intersection point.
4. A system of two linear equations in three variables for which the solution set is

$$L = \{(2t, t + 1, 3 - t) : t \in \mathbb{R}\}.$$

5. (Challenge) Four distinct lines in $\mathbb{R}^2$ such that any three of the lines have a common intersection point, but the four lines have no common intersection point.

PROBLEM 1.2

When presented with a system of equations, the mathematician's first urge is to find the solutions. However, solving equations takes time, effort, and, if there are a lot of equations, it takes money.

Sometimes we can learn valuable information about the solutions of an equation without actually solving it. As they say, the fastest (and cheapest) computation is the one you never do.

For systems of linear equations, there are three qualitatively different possibilities for the number of solutions: there will always be either **infinitely many**, **one**, or **none**. In this problem we develop a handy heuristic for determining at a glance which of these qualitative regimes we expect to be in based on the number of variables and the number of equations in our system.

1. Let $n \geq 1$. How we many solutions should we expect a system of n linear equations in two variables to have? Explain why this is the case.

2. Same question but now with equations in three variables.

3. If you had a system of linear equations in 1000 variables and 850 equations, how many solutions do you expect the system to have and why?

4. Extrapolating from the previous two problems, make a guess about what the answer should for equations in $m \geq 4$ variables.