MATH 221 · FALL 2026

Problem Set 3: Row Reduction, Vectors, and Span

DUE Friday, September 18th by 11:59pm

Alice laughed. "There's no use trying," she said. "One can't believe impossible things." "I daresay you haven't had much practice", said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

From *Alice in Wonderland* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

---

PROBLEM 3.1

When potassium permanganate ($KMnO_4$) is mixed with hydrochloric acid (HCl), the result is potassium chloride (KCl), manganese chloride ($MnCl_2$), water (H_2O), and chlorine gas (Cl_2):

$$x_1\,KMnO_4 + x_2\,HCl \longrightarrow x_3\,KCl + x_4\,MnCl_2 + x_5\,H_2O + x_6\,Cl_2.$$

This is a standard way of producing chlorine gas in a lab. *Do not try it at home.*

1. Before we set out trying to solve for the x_i, how many solutions should we expect to find? Explain.

2. Write down the system of linear equations that $x_1, \ldots, x_6$ must satisfy in order for this reaction to be balanced. Use row reduction to find the general solution to your linear system. You may use the Row Reducer to help you calculate.

3. A chemist wants the balanced reaction with the smallest possible positive whole number coefficients. Help them find this solution.

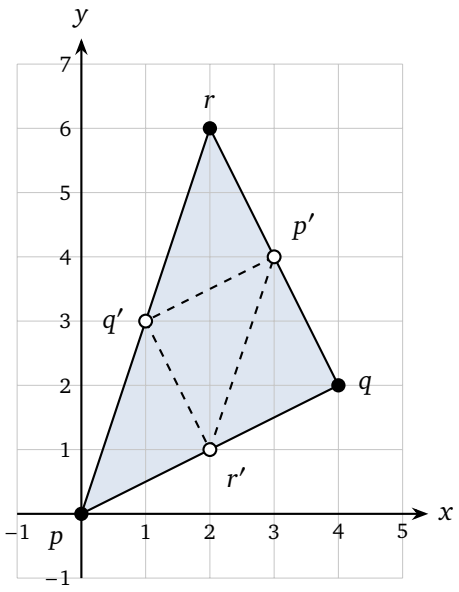
4. (Challenge) Carbon monoxide, carbon dioxide, and hydrogen gas can react to produce methane and water:

   $$x_1\,CO + x_2\,CO_2 + x_3\,H_2 \longrightarrow x_4\,CH_4 + x_5\,H_2O.$$

   Find the general solution to the system of equations for this reaction. You should find two free variables. Explain what this means chemically: show that every balanced version of this reaction is built out of two simpler reactions, and find those two reactions.

PROBLEM 3.2

Given a triangle with vertices p, q, r, the **medial triangle** is the triangle whose vertices are the midpoints of the three sides. We label the vertices of the medial triangle p', q', r' so that p' is the midpoint of the side opposite p, q' is the midpoint of the side opposite q, and r' is the midpoint of the side opposite r. See the picture below for an example: the solid dots are the vertices of the triangle and the hollow dots are the vertices of the medial triangle.



1. Let $p = (-2, 1)$, $q = (4, -1)$, and $r = (2, 5)$. Find the vertices p', q', r' of the medial triangle. Note that these are **not** the same points as in the picture.

2. Now suppose you are only told the vertices of the medial triangle: $p' = (1, 4)$, $q' = (-2, 1)$, and $r' = (3, 0)$. Find the vertices p, q, r of the original triangle.

3. Now let p, q, r be any triangle and let p', q', r' be the vertices of its medial triangle. Find formulas for p, q, r in terms of p', q', r'.

PROBLEM 3.3

Recall that the **span** of a collection of vectors $v_1, \ldots, v_k$ is the set

$$\text{Span}\{v_1, \ldots, v_k\} := \{c_1 v_1 + \cdots + c_k v_k : c_i \in \mathbb{R}\}$$

of all their linear combinations. For each of the following, determine whether b is in $\text{Span}\{v_1, \ldots, v_k\}$. If it is, write b explicitly as a linear combination of the v_i. If it is not, explain how you know.

1. $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

2. $v_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

3. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

4. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}$

PROBLEM 3.4

The Row Reducer is happy to help you with arithmetic, but it only works when the matrix is filled with actual numbers. In this problem some of the entries are an unknown constant c, and that means you get to do the row reduction by hand.

Consider the system

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x - 3y + 4z &= 3 \\ 3x - 5y + cz &= 5 \end{aligned}$$

where c is a real number. For which values of c does the system have no solutions? Exactly one solution? Infinitely many solutions? **You do not need to find the exact solutions.**