MATH 221 · FALL 2026

Problem Set 3: Row Reduction, Vectors, and Span

SOLUTIONS

Alice laughed. "There's no use trying," she said. "One can't believe impossible things." "I daresay you haven't had much practice", said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

From *Alice in Wonderland* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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PROBLEM 3.1

When potassium permanganate ($KMnO_4$) is mixed with hydrochloric acid (HCl), the result is potassium chloride (KCl), manganese chloride ($MnCl_2$), water (H_2O), and chlorine gas (Cl_2):

$$x_1\,KMnO_4 + x_2\,HCl \longrightarrow x_3\,KCl + x_4\,MnCl_2 + x_5\,H_2O + x_6\,Cl_2.$$

This is a standard way of producing chlorine gas in a lab. *Do not try it at home.*

1. Before we set out trying to solve for the x_i, how many solutions should we expect to find? Explain.

   SOLUTION

   Assuming this is a valid chemical reaction, there should be infinitely many possible solutions. The reason is that given any solution, we can scale up the recipe by an arbitrary factor and get another solution. There's an infinite number of possible scalings, hence infinitely many possible solutions. □

2. Write down the system of linear equations that $x_1, \ldots, x_6$ must satisfy in order for this reaction to be balanced. Use row reduction to find the general solution to your linear system. You may use the Row Reducer to help you calculate.

SOLUTION

There are five elements involved in the reaction, and each one gives us an equation by counting the atoms on each side of the arrow:

$$\begin{aligned} \text{K:} \quad & x_1 = x_3 \\ \text{Mn:} \quad & x_1 = x_4 \\ \text{O:} \quad & 4x_1 = x_5 \\ \text{H:} \quad & x_2 = 2x_5 \\ \text{Cl:} \quad & x_2 = x_3 + 2x_4 + 2x_6. \end{aligned}$$

Moving every variable to the left side gives us a homogeneous system of equations represented by the following matrix,

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & -2 & 0 & -2 \end{bmatrix}.$$

Row reducing we end up with

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{16}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{8}{5} & 0 \end{array}\right].$$

The sixth column has no pivot, so x_6 is a free variable. Setting $x_6 = t$ and solving for the dependent variables gives the general solution

$$(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\tfrac{2}{5}t,\ \tfrac{16}{5}t,\ \tfrac{2}{5}t,\ \tfrac{2}{5}t,\ \tfrac{8}{5}t,\ t\right). \qquad \square$$

3. A chemist wants the balanced reaction with the smallest possible positive whole number coefficients. Help them find this solution.

SOLUTION

All the fractions have a 5 in the denominator, so we need t to be a positive multiple of 5 in order to clear all the denominators. Setting $t = 5$ then gives us the solution we are looking for: $(x_1, \ldots, x_6) = (2, 16, 2, 2, 8, 5)$, or in other words

$$2\,\mathrm{KMnO_4} + 16\,\mathrm{HCl} \longrightarrow 2\,\mathrm{KCl} + 2\,\mathrm{MnCl_2} + 8\,\mathrm{H_2O} + 5\,\mathrm{Cl_2}.$$

SOLUTION (CONTINUED)

Every other balanced version of the reaction is a whole number multiple of this one. □

4. (Challenge) Carbon monoxide, carbon dioxide, and hydrogen gas can react to produce methane and water:

$$x_1\,\mathrm{CO} + x_2\,\mathrm{CO_2} + x_3\,\mathrm{H_2} \longrightarrow x_4\,\mathrm{CH_4} + x_5\,\mathrm{H_2O}.$$

Find the general solution to the system of equations for this reaction. You should find two free variables. Explain what this means chemically: show that every balanced version of this reaction is built out of two simpler reactions, and find those two reactions.

SOLUTION

Counting carbon, oxygen, and hydrogen atoms gives the system

$$\begin{aligned} \text{C:}\quad & x_1 + x_2 = x_4 \\ \text{O:}\quad & x_1 + 2x_2 = x_5 \\ \text{H:}\quad & 2x_3 = 4x_4 + 2x_5. \end{aligned}$$

We convert this system to an augmented matrix, and then row reduce.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{2} R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right]$$

Columns 4 and 5 have no pivots, so x_4 and x_5 are free. Setting $x_4 = s$ and $x_5 = t$ we get the general solution

$$(x_1, x_2, x_3, x_4, x_5) = (2s - t,\ t - s,\ 2s + t,\ s,\ t).$$

To interpret this chemically, let's look for the simplest solutions with whole number coefficients that are not negative. Taking $(s, t) = (1, 1)$ gives $(1, 0, 3, 1, 1)$ and taking $(s, t) = (1, 2)$ gives $(0, 1, 4, 1, 2)$. These correspond to the two reactions

$$\begin{aligned} \mathrm{CO} + 3\,\mathrm{H_2} &\longrightarrow \mathrm{CH_4} + \mathrm{H_2O} \\ \mathrm{CO_2} + 4\,\mathrm{H_2} &\longrightarrow \mathrm{CH_4} + 2\,\mathrm{H_2O}, \end{aligned}$$

each of which is balanced on its own. The first uses only carbon monoxide and the second uses only carbon dioxide.

Now suppose we run the first reaction a times and the second reaction b times. Adding up all the molecules involved, we use a molecules of CO, b molecules of $\mathrm{CO_2}$, and $3a + 4b$

SOLUTION (CONTINUED)

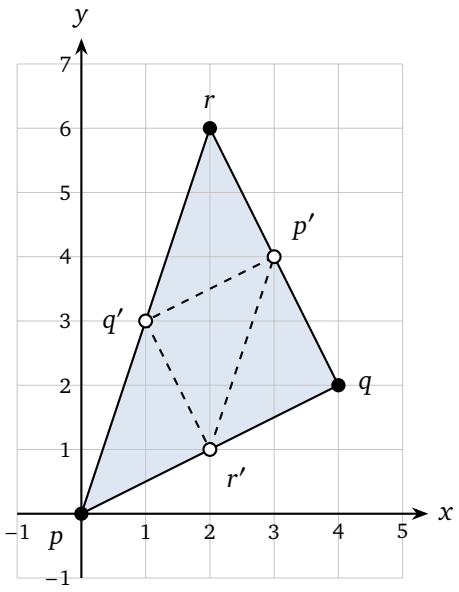
molecules of H_2, and we produce $a+b$ molecules of CH_4 and $a+2b$ molecules of H_2O. If we set $a = 2s - t$ and $b = t - s$, then

$$\begin{aligned} 3a + 4b &= 3(2s - t) + 4(t - s) = 2s + t \\ a + b &= (2s - t) + (t - s) = s \\ a + 2b &= (2s - t) + 2(t - s) = t, \end{aligned}$$

which is exactly the general solution we found above. So every balanced version of the original reaction is what you get by running the carbon monoxide reaction some number of times and the carbon dioxide reaction some number of times. The two free variables are telling us that there are really two independent reactions hiding inside the one we wrote down. □

PROBLEM 3.2

Given a triangle with vertices p, q, r, the **medial triangle** is the triangle whose vertices are the midpoints of the three sides. We label the vertices of the medial triangle p', q', r' so that p' is the midpoint of the side opposite p, q' is the midpoint of the side opposite q, and r' is the midpoint of the side opposite r. See the picture below for an example: the solid dots are the vertices of the triangle and the hollow dots are the vertices of the medial triangle.



1. Let $p = (-2, 1)$, $q = (4, -1)$, and $r = (2, 5)$. Find the vertices p', q', r' of the medial triangle. Note that these are **not** the same points as in the picture.

SOLUTION

The midpoint of two points u and v is $\frac{1}{2}(u+v)$. Therefore

$$\begin{aligned} p' &= \tfrac{1}{2}(q+r) = \tfrac{1}{2}\big((4,-1)+(2,5)\big) = \tfrac{1}{2}(6,4) = (3,2) \\ q' &= \tfrac{1}{2}(p+r) = \tfrac{1}{2}\big((-2,1)+(2,5)\big) = \tfrac{1}{2}(0,6) = (0,3) \\ r' &= \tfrac{1}{2}(p+q) = \tfrac{1}{2}\big((-2,1)+(4,-1)\big) = \tfrac{1}{2}(2,0) = (1,0). \end{aligned}$$

□

2. Now suppose you are only told the vertices of the medial triangle: $p' = (1,4)$, $q' = (-2,1)$, and $r' = (3,0)$. Find the vertices p, q, r of the original triangle.

SOLUTION

Write $p = (p_1, p_2)$, $q = (q_1, q_2)$, and $r = (r_1, r_2)$. Since p' is the midpoint of q and r we have $\frac{1}{2}(q+r) = (1,4)$, and similarly $\frac{1}{2}(p+r) = (-2,1)$ and $\frac{1}{2}(p+q) = (3,0)$. Multiplying through by 2 and looking at the first coordinates gives the linear system

$$\begin{aligned} q_1 + r_1 &= 2 \\ p_1 + r_1 &= -4 \\ p_1 + q_1 &= 6, \end{aligned}$$

and the second coordinates give the same system with a different right hand side:

$$\begin{aligned} q_2 + r_2 &= 8 \\ p_2 + r_2 &= 2 \\ p_2 + q_2 &= 0. \end{aligned}$$

Since the two systems have the same coefficients, we can row reduce them at the same time by using two augmented columns.

$$\left[\begin{array}{ccc|cc} 0 & 1 & 1 & 2 & 8 \\ 1 & 0 & 1 & -4 & 2 \\ 1 & 1 & 0 & 6 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 1 & 2 & 8 \\ 1 & 1 & 0 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 1 & -1 & 10 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 0 & -2 & 8 & -10 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{2} R_3} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 0 & 1 & -4 & 5 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 1 & 2 & 8 \\ 0 & 0 & 1 & -4 & 5 \end{array}\right]$$

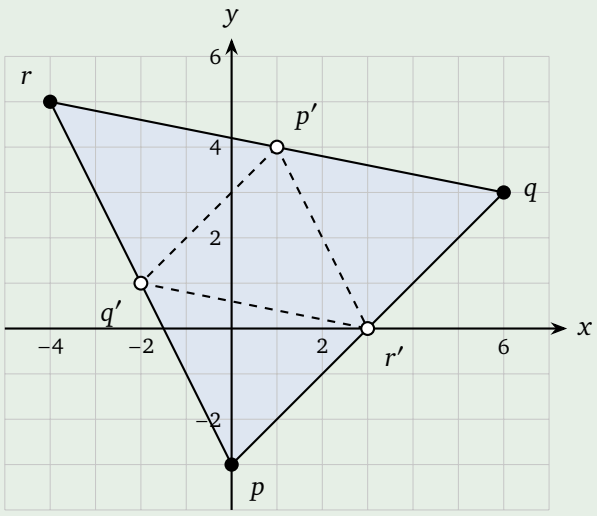
$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 6 & 3 \\ 0 & 0 & 1 & -4 & 5 \end{array}\right]$$

SOLUTION (CONTINUED)

Reading off the two augmented columns, $(p_1, q_1, r_1) = (0, 6, -4)$ and $(p_2, q_2, r_2) = (-3, 3, 5)$. So the original triangle has vertices

$$p = (0, -3), \qquad q = (6, 3), \qquad r = (-4, 5).$$

Here is the picture.



□

3. Now let p, q, r be any triangle and let p', q', r' be the vertices of its medial triangle. Find formulas for p, q, r in terms of p', q', r'.

SOLUTION

By definition of the medial triangle,

$$p' = \tfrac{1}{2}(q + r), \qquad q' = \tfrac{1}{2}(p + r), \qquad r' = \tfrac{1}{2}(p + q).$$

The point p appears in q' and r' but not in p', so we add the two that contain it and subtract the one that does not:

$$q' + r' - p' = \tfrac{1}{2}(p + r) + \tfrac{1}{2}(p + q) - \tfrac{1}{2}(q + r) = p.$$

The same trick recovers the other two vertices, so

$$p = q' + r' - p', \qquad q = p' + r' - q', \qquad r = p' + q' - r'.$$

Geometrically, the formula $p = q' + r' - p'$ says that p is the fourth vertex of the parallelogram with vertices q', p', r' in which p and p' are opposite. You can see this parallelogram in either of the pictures above. □

PROBLEM 3.3

Recall that the **span** of a collection of vectors $v_1, \ldots, v_k$ is the set

$$\text{Span}\{v_1, \ldots, v_k\} := \{c_1 v_1 + \cdots + c_k v_k : c_i \in \mathbb{R}\}$$

of all their linear combinations. For each of the following, determine whether b is in $\text{Span}\{v_1, \ldots, v_k\}$. If it is, write b explicitly as a linear combination of the v_i. If it is not, explain how you know.

1. $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

SOLUTION

The vector equation $x_1 v_1 + x_2 v_2 = b$ is equivalent to the linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 - x_2 &= 3 \\ -x_1 + 3x_2 &= 1, \end{aligned}$$

whose augmented matrix has v_1, v_2 as its first two columns and b as its augmented column. We row reduce.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -1 & 3 \\ -1 & 3 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -5 \\ -1 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{5} R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent with the unique solution $(x_1, x_2) = (2, 1)$. So b is in the span, and in fact $b = 2v_1 + v_2$. □

2. $v_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

SOLUTION

We set up the augmented matrix $[\, v_1 \; v_2 \mid b \,]$ and row reduce. Swapping the third row to the top first avoids some fractions early on, but they show up eventually either way.

SOLUTION (CONTINUED)

$$\left[\begin{array}{cc|c} 2 & 4 & 1 \\ 3 & 1 & 4 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -8 & 4 \\ 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -8 & 4 \\ 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{8}R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -2 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent with the unique solution $(x_1, x_2) = \left(\frac{3}{2}, -\frac{1}{2}\right)$, so b is in the span:

$$b = \tfrac{3}{2}v_1 - \tfrac{1}{2}v_2.$$

There is nothing wrong with fractional coefficients. You can check directly that $\frac{3}{2}(2,3,1) - \frac{1}{2}(4,1,3) = (3-2,\ \frac{9}{2} - \frac{1}{2},\ \frac{3}{2} - \frac{3}{2}) = (1,4,0)$. □

3. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

SOLUTION

Now there are three unknowns x_1, x_2, x_3 and four equations. We row reduce the augmented matrix $[\, v_1\ v_2\ v_3 \mid b \,]$.

SOLUTION (CONTINUED)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 3 & 5 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is no contradictory row, so the system is consistent and b is in the span. But the third column has no pivot, so x_3 is free and there are infinitely many ways to write b as a linear combination. Setting $x_3 = t$ we get $x_1 = 2 - t$ and $x_2 = 1 - t$, so

$$b = (2 - t)\, v_1 + (1 - t)\, v_2 + t\, v_3 \qquad \text{for any } t \in \mathbb{R}.$$

For instance $t = 0$ gives $b = 2v_1 + v_2$, and $t = 1$ gives $b = v_1 + v_3$. □

4. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}$

SOLUTION

The vectors are the same as in the previous part and only the last coordinate of b has changed, so the row reduction goes exactly the same way.

SOLUTION (CONTINUED)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 5 & 5 \\ 1 & -1 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

The last row says $0 = 1$, so the system is inconsistent and b is **not** in the span. □

PROBLEM 3.4

The Row Reducer is happy to help you with arithmetic, but it only works when the matrix is filled with actual numbers. In this problem some of the entries are an unknown constant c, and that means you get to do the row reduction by hand.

Consider the system

$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x - 3y + 4z &= 3 \\ 3x - 5y + cz &= 5 \end{aligned}$$

where c is a real number. For which values of c does the system have no solutions? Exactly one solution? Infinitely many solutions? **You do not need to find the exact solutions.**

SOLUTION

We row reduce the augmented matrix, treating c as a fixed but unknown number.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -3 & 4 & 3 \\ 3 & -5 & c & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 3 & -5 & c & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & c-9 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & c-7 & 1 \end{array}\right]$$

The last row is the equation $(c - 7)z = 1$. We have to consider some cases:

- If $c = 7$, this row says $0 = 1$, so the system has **no solutions**.

SOLUTION (CONTINUED)

- If $c \neq 7$, then $c - 7$ is a nonzero number and we are allowed to divide by it. Every column then has a pivot, so there is **exactly one solution**.
- There is no value of c for which the system has **infinitely many solutions**. □