MATH 221 · FALL 2026

Problem Set 4: Solution Sets and Linear Independence

DUE Friday, October 2nd by 11:59pm

Tiger got to hunt, bird got to fly;
Man got to sit and wonder 'why, why, why?'
Tiger got to sleep, bird got to land;
Man got to tell himself he understand.

From *Cat's Cradle* by Kurt Vonnegut

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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PROBLEM 4.1

In class we tried to recover a 2×2 matrix from its row sums and column sums. Let's try the same thing with a 3×3 matrix. Suppose the matrix below has the row sums and column sums written along its right side and bottom.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{matrix} 10 \\ 6 \\ 8 \end{matrix}$$
$$\begin{matrix} 7 & 9 & 8 \end{matrix}$$

1. Before we set out to find the x_i, explain why the row sums of any matrix must add up to the same number as its column sums. What does this tell you about the number of solutions if the column sums had been $7, 9, 9$ instead?

2. Write down the system of six linear equations that $x_1, \ldots, x_9$ must satisfy, and use row reduction to find the general solution. Write your answer in parametric vector form (that is, in the form $v_0 + t_1 v_1 + \cdots + t_k v_k$, where v_0 is a particular solution and there is one parameter t_i for each free variable). You may use the Row Reducer to help you calculate.

3. Your answer to part (b) has the form $v_0 + \text{Span}\{v_1, \ldots, v_k\}$. Arrange the entries of v_0 and of each v_i back into a 3×3 grid. What do you notice about the v_i?

4. There are other ways to place the 2×2 pattern in a 3×3 grid, for example in the bottom right corner:

$$w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

   Write w as a linear combination of the vectors $v_1, \ldots, v_k$ you found in part (c).

5. (Challenge) The free variables in part (b) were x_5, x_6, x_8, x_9, which are exactly the entries in the bottom right 2×2 block of the grid. Explain, without row reducing, why you can fill in that block with any four numbers you like and the remaining five entries are then completely determined. How many free variables would you expect if we replaced 3×3 by $n \times n$?

PROBLEM 4.2

For each of the following collections of vectors, find the dimension of $\text{Span}\{v_1, \ldots, v_k\}$ and find a minimal collection of the v_i which has the same span.

1. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$

2. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

3. $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$

PROBLEM 4.3

A four-legged table placed on a flat floor wobbles unless the ends of its four legs all lie on a common plane. If the ends of the legs are at the points p_1, p_2, p_3, p_4 in $\mathbb{R}^3$, then the four points lie on a common plane exactly when the three vectors

$$p_2 - p_1, \qquad p_3 - p_1, \qquad p_4 - p_1$$

lie on a common plane through the origin, and that happens exactly when the three vectors are linearly dependent.

1. The legs of a table end at the points $p_1 = (0, 0, 0)$, $p_2 = (6, 0, 1)$, $p_3 = (0, 4, 2)$, and $p_4 = (6, 4, 3)$. Does the table wobble?

2. A second table has legs ending at the points $p_1 = (1, 1, 0)$, $p_2 = (7, 1, 2)$, $p_3 = (1, 5, 1)$, and $p_4 = (7, 5, 4)$. Does this table wobble?

3. A carpenter decides to fix the second table by adjusting the length of its fourth leg, so that the end of that leg moves to $(7, 5, c)$. For which value of c will the table stop wobbling?

4. (Challenge) Explain why a three-legged stool never wobbles, no matter how uneven its legs are.