MATH 221 · FALL 2026

# Problem Set 4: Solution Sets and Linear Independence

**SOLUTIONS**

*Tiger got to hunt, bird got to fly;*
*Man got to sit and wonder 'why, why, why?'*
*Tiger got to sleep, bird got to land;*
*Man got to tell himself he understand.*

From *Cat's Cradle* by Kurt Vonnegut

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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## PROBLEM 4.1

In class we tried to recover a $2 \times 2$ matrix from its row sums and column sums. Let's try the same thing with a $3 \times 3$ matrix. Suppose the matrix below has the row sums and column sums written along its right side and bottom.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{matrix} 10 \\ 6 \\ 8 \end{matrix}$$

$$\begin{matrix} 7 & 9 & 8 \end{matrix}$$

1. Before we set out to find the $x_i$, explain why the row sums of any matrix must add up to the same number as its column sums. What does this tell you about the number of solutions if the column sums had been $7, 9, 9$ instead?

**SOLUTION**

Adding up the row sums counts every entry of the matrix exactly once, and so does adding up the column sums. Both give the total of all the entries, so they must agree. In our matrix both add up to $10 + 6 + 8 = 7 + 9 + 8 = 24$.
If the column sums were $7, 9, 9$, they would add up to 25 while the row sums add up to 24. No matrix can have these row and column sums, so the system would have no solutions. □

2. Write down the system of six linear equations that $x_1, \ldots, x_9$ must satisfy, and use row reduction to find the general solution. Write your answer in parametric vector form (that is, in the form $v_0 + t_1 v_1 + \cdots + t_k v_k$, where $v_0$ is a particular solution and there is one parameter $t_i$ for each free variable). You may use the Row Reducer to help you calculate.

**SOLUTION**

The three row sums and three column sums give the equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 10 & \qquad x_1 + x_4 + x_7 &= 7 \\ x_4 + x_5 + x_6 &= 6 & \qquad x_2 + x_5 + x_8 &= 9 \\ x_7 + x_8 + x_9 &= 8 & \qquad x_3 + x_6 + x_9 &= 8. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 8 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 8 \end{array}\right].$$

Row reducing we end up with

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -7 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The row of zeros at the bottom is the equation $0 = 0$: one of our six equations was a consequence of the other five. Columns 5, 6, 8, 9 have no pivots, so $x_5, x_6, x_8, x_9$ are free. Setting $x_5 = t_1$, $x_6 = t_2$, $x_8 = t_3$, $x_9 = t_4$ and solving for the other variables gives

$$\begin{aligned} x_1 &= -7 + t_1 + t_2 + t_3 + t_4 \\ x_2 &= 9 - t_1 - t_3 \\ x_3 &= 8 - t_2 - t_4 \\ x_4 &= 6 - t_1 - t_2 \\ x_7 &= 8 - t_3 - t_4. \end{aligned}$$

**SOLUTION (CONTINUED)**

Writing the unknowns back in their places in the matrix, the general solution in parametric vector form is

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} = \begin{bmatrix} -7 & 9 & 8 \\ 6 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} + t_1 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} + t_4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

for any $t_1, t_2, t_3, t_4 \in \mathbb{R}$. □

3. Your answer to part (b) has the form $v_0 + \text{Span}\{v_1, \ldots, v_k\}$. Arrange the entries of $v_0$ and of each $v_i$ back into a $3 \times 3$ grid. What do you notice about the $v_i$?

**SOLUTION**

Reading off the answer to part (b), the particular solution $v_0$ is the matrix

$$\left[\begin{array}{ccc|c} -7 & 9 & 8 & 10 \\ 6 & 0 & 0 & 6 \\ 8 & 0 & 0 & 8 \\ \hline 7 & 9 & 8 & \end{array}\right]$$

which does have the required row sums and column sums, and the four vectors are

$$v_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Each $v_i$ is the $2 \times 2$ pattern

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

from the $2 \times 2$ case, placed in two of the rows and two of the columns of the grid, with zeros everywhere else. The four $v_i$ are the four ways to place the pattern so that it uses the first row and the first column. Every row and every column of each $v_i$ adds up to 0, which is exactly why adding a multiple of $v_i$ to a solution does not change the row sums or the column sums. □

4. There are other ways to place the $2 \times 2$ pattern in a $3 \times 3$ grid, for example in the bottom right corner:

$$w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Write $w$ as a linear combination of the vectors $v_1, \ldots, v_k$ you found in part (c).

**SOLUTION**

Every row and every column of $w$ adds up to 0, so $w$ is a solution of the homogeneous system, and we expect it to be a linear combination of $v_1, v_2, v_3, v_4$. To write $w = y_1v_1 + y_2v_2 + y_3v_3 + y_4v_4$, look at the four entries in the bottom right block. The entry in row 2, column 2 is 1 in $v_1$ and 0 in the other three $v_i$, so $y_1 = 1$. In the same way, the entries in row 2, column 3 and in row 3, column 2 force $y_2 = -1$ and $y_3 = -1$, and the entry in row 3, column 3 forces $y_4 = 1$. Checking the remaining entries,

$$v_1 - v_2 - v_3 + v_4 = \begin{bmatrix} 1-1-1+1 & -1+0+1+0 & 0+1+0-1 \\ -1+1+0+0 & 1 & -1 \\ 0+0+1-1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = w.$$

So $w$ is in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$: adding a multiple of $w$ to a solution gives another solution, and the general solution in part (b) already accounts for this. □

5. (Challenge) The free variables in part (b) were $x_5, x_6, x_8, x_9$, which are exactly the entries in the bottom right $2 \times 2$ block of the grid. Explain, without row reducing, why you can fill in that block with any four numbers you like and the remaining five entries are then completely determined. How many free variables would you expect if we replaced $3 \times 3$ by $n \times n$?

**SOLUTION**

Suppose we have chosen $x_5, x_6, x_8, x_9$. The second and third row sums force $x_4 = 6 - x_5 - x_6$ and $x_7 = 8 - x_8 - x_9$, and the second and third column sums force $x_2 = 9 - x_5 - x_8$ and $x_3 = 8 - x_6 - x_9$. That leaves $x_1$, which appears in both the first row sum and the first column sum. The first row sum says

$$x_1 = 10 - x_2 - x_3 = 10 - (9 - x_5 - x_8) - (8 - x_6 - x_9) = -7 + x_5 + x_6 + x_8 + x_9,$$

and the first column sum says

$$x_1 = 7 - x_4 - x_7 = 7 - (6 - x_5 - x_6) - (8 - x_8 - x_9) = -7 + x_5 + x_6 + x_8 + x_9.$$

These agree, so there is exactly one way to finish filling in the grid. The two formulas for $x_1$ agree precisely because the row sums and the column sums have the same total, which is what we checked in part (a).

The same argument works for an $n \times n$ grid. Fill in the bottom right $(n-1) \times (n-1)$ block however you like. Then rows 2 through $n$ determine the rest of the first column, columns 2 through $n$ determine the rest of the first row, and the top left entry is determined by the first row (or the first column, and these agree because the totals agree). So we expect $(n-1)^2$ free variables. When $n = 2$ this is 1, matching the single pattern we found in class, and when $n = 3$ it is 4, matching part (b). □

## PROBLEM 4.2

For each of the following collections of vectors, find the dimension of $\text{Span}\{v_1, \ldots, v_k\}$ and find a minimal collection of the $v_i$ which has the same span.

1. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$

### SOLUTION

We row reduce the matrix whose columns are $v_1, v_2, v_3$.

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & 1 & -3 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{5}{4} R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{4} R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns have pivots and the third does not. The third column of the reduced matrix tells us $v_3 = 2v_1 - v_2$, so $v_3$ is in $\text{Span}\{v_1, v_2\}$ and removing it does not change the span. The pivot columns $v_1$ and $v_2$ are linearly independent, so $v_1, v_2$ is a linearly independent collection with $\text{Span}\{v_1, v_2\} = \text{Span}\{v_1, v_2, v_3\}$, and the dimension is 2. □

2. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

### SOLUTION

We row reduce the matrix whose columns are $v_1, v_2, v_3, v_4$.

**SOLUTION (CONTINUED)**

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - 2R_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_4 \to \frac{1}{3} R_4} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_4} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_4} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Every column has a pivot, so $v_1, v_2, v_3, v_4$ are already linearly independent and $\text{Span}\{v_1, v_2, v_3, v_4\}$ has dimension 4. The whole collection is the linearly independent collection we are looking for. □

3. $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \end{bmatrix}$

**SOLUTION**

We row reduce the matrix whose columns are $v_1, v_2, v_3, v_4$.

### SOLUTION (CONTINUED)

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 1 & 5 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only the first two columns have pivots. The last two columns of the reduced matrix tell us $v_3 = v_1 - v_2$ and $v_4 = 2v_1 + v_2$, so removing $v_3$ and $v_4$ does not change the span. The pivot columns $v_1$ and $v_2$ are linearly independent, so $v_1, v_2$ is a linearly independent collection with $\text{Span}\{v_1, v_2\} = \text{Span}\{v_1, v_2, v_3, v_4\}$, and the dimension is 2. □

## PROBLEM 4.3

A four-legged table placed on a flat floor wobbles unless the ends of its four legs all lie on a common plane. If the ends of the legs are at the points $p_1, p_2, p_3, p_4$ in $\mathbb{R}^3$, then the four points lie on a common plane exactly when the three vectors

$$p_2 - p_1, \qquad p_3 - p_1, \qquad p_4 - p_1$$

lie on a common plane through the origin, and that happens exactly when the three vectors are linearly dependent.

1. The legs of a table end at the points $p_1 = (0, 0, 0)$, $p_2 = (6, 0, 1)$, $p_3 = (0, 4, 2)$, and $p_4 = (6, 4, 3)$. Does the table wobble?

### SOLUTION

The three vectors are

$$p_2 - p_1 = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}, \qquad p_3 - p_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \qquad p_4 - p_1 = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}.$$

We row reduce the matrix with these vectors as its columns.

**SOLUTION (CONTINUED)**

$$\begin{bmatrix} 6 & 0 & 6 \\ 0 & 4 & 4 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 6 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 6R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & -12 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column has no pivot, so the three vectors are linearly dependent. In fact the reduced matrix tells us $p_4 - p_1 = (p_2 - p_1) + (p_3 - p_1)$. The ends of the legs lie on a common plane, so the table does **not** wobble. □

2. A second table has legs ending at the points $p_1 = (1, 1, 0)$, $p_2 = (7, 1, 2)$, $p_3 = (1, 5, 1)$, and $p_4 = (7, 5, 4)$. Does this table wobble?

**SOLUTION**

This time the three vectors are

$$p_2 - p_1 = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}, \qquad p_3 - p_1 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \qquad p_4 - p_1 = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix},$$

and we row reduce the matrix with these columns.

$$\begin{bmatrix} 6 & 0 & 6 \\ 0 & 4 & 4 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_1} \begin{bmatrix} 6 & 0 & 6 \\ 0 & 4 & 4 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{6}R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -\frac{1}{4}R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**SOLUTION (CONTINUED)**

Every column has a pivot, so the three vectors are linearly independent. The ends of the legs do not lie on a common plane, so this table **wobbles**. □

3. A carpenter decides to fix the second table by adjusting the length of its fourth leg, so that the end of that leg moves to $(7, 5, c)$. For which value of $c$ will the table stop wobbling?

**SOLUTION**

Now $p_4 - p_1 = (6, 4, c)$ while the other two vectors are unchanged, so we row reduce the same matrix as in part (b) with the 4 in the bottom right corner replaced by $c$. Since $c$ is unknown we do the row reduction by hand.

$$\begin{bmatrix} 6 & 0 & 6 \\ 0 & 4 & 4 \\ 2 & 1 & c \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{6} R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 2 & 1 & c \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{4} R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & c \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & c-2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & c-3 \end{bmatrix}$$

If $c \neq 3$, the bottom right entry is nonzero and every column has a pivot, so the vectors are linearly independent and the table still wobbles. If $c = 3$, the third column has no pivot, the vectors are linearly dependent, and the ends of the legs lie on a common plane. So the carpenter should adjust the fourth leg so that it ends at $(7, 5, 3)$. □

4. (Challenge) Explain why a three-legged stool never wobbles, no matter how uneven its legs are.

**SOLUTION**

Suppose the legs end at $p_1, p_2, p_3$. The stool does not wobble as long as these three points lie on a common plane, which happens exactly when the two vectors $p_2 - p_1$ and $p_3 - p_1$ lie on a common plane through the origin. Two vectors always do. If they are linearly independent, then $\text{Span}\{p_2 - p_1, p_3 - p_1\}$ is a plane through the origin containing both of them. If they are linearly dependent, then they lie on a common line through the origin, and any plane containing that line works. Either way the three points lie on a common plane, so the stool stands flat. □