MATH 221 · FALL 2026

Problem Set 5: Subspaces, Bases, and Dimension

SOLUTIONS

I call our world Flatland, not because we call it so, but to make its nature clearer to you, my happy readers, who are privileged to live in Space.

From *Flatland* by Edwin A. Abbott

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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PROBLEM 5.1

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & -5 \end{bmatrix}.$$

You may use the Row Reducer to help you calculate.

1. Find a basis for $\text{Nul}(A)$. What is $\dim \text{Nul}(A)$?

SOLUTION

$\text{Nul}(A)$ is the set of solutions of the homogeneous system $Ax = 0$, so we row reduce A.

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & -5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 2 & -1 & 5 & -5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 3 and 4 have no pivots, so x_3 and x_4 are free, and the reduced matrix says

SOLUTION (CONTINUED)

$x_1 = -2x_3 + x_4$ and $x_2 = x_3 - 3x_4$. The general solution in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

So $\text{Nul}(A) = \text{Span}\{v_1, v_2\}$ where

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

These two vectors are linearly independent, since neither is a multiple of the other (look at the last two entries), so v_1, v_2 is a basis for $\text{Nul}(A)$ and $\dim \text{Nul}(A) = 2$. □

2. Find a basis for $\text{Col}(A)$. What is $\dim \text{Col}(A)$?

SOLUTION

The pivot columns of A form a basis for the column space $\text{Col}(A)$. From the row reduction in part (a), the pivots are in columns 1 and 2, so a basis for $\text{Col}(A)$ consists of the first two columns of A,

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and thus $\dim \text{Col}(A) = 2$. □

3. Is the vector $b = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$ in $\text{Col}(A)$?

SOLUTION

The vector b belongs to $\text{Col}(A)$ exactly when the system $Ax = b$ has a solution, so we row reduce the augmented matrix.

SOLUTION (CONTINUED)

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 1 & 1 & 1 & 2 & -1 \\ 2 & -1 & 5 & -5 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 3 & -3 \\ 2 & -1 & 5 & -5 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 3 & -3 \\ 0 & -1 & 1 & -3 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There is no pivot in the augmented column, so the system is consistent and thus b does belong to $\mathrm{Col}(A)$. Taking $x_3 = x_4 = 0$ gives the solution $x_1 = 2$, $x_2 = -3$, so

$$b = 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

□

4. Is the vector $v = \begin{bmatrix} -3 \\ 4 \\ 1 \\ -1 \end{bmatrix}$ in $\mathrm{Nul}(A)$?

SOLUTION

The vector v is in $\mathrm{Nul}(A)$ exactly when $Av = 0$, and

$$Av = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & -5 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3+0+2+1 \\ -3+4+1-2 \\ -6-4+5+5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So v **is** in $\mathrm{Nul}(A)$. □

5. (Challenge) Find an equation in b_1, b_2, b_3 that is satisfied exactly when the vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ belongs to the column space $\mathrm{Col}(A)$.

SOLUTION

We row reduce the augmented matrix $[\,A \mid b\,]$ with a general right hand side, following

SOLUTION (CONTINUED)

the same steps as in part (c).

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & b_1 \\ 1 & 1 & 1 & 2 & b_2 \\ 2 & -1 & 5 & -5 & b_3 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & b_1 \\ 0 & 1 & -1 & 3 & b_2 - b_1 \\ 2 & -1 & 5 & -5 & b_3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & b_1 \\ 0 & 1 & -1 & 3 & b_2 - b_1 \\ 0 & -1 & 1 & -3 & b_3 - 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & b_1 \\ 0 & 1 & -1 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 3b_1 \end{array}\right]$$

The last row is the equation $0 = b_2 + b_3 - 3b_1$. The system is consistent exactly when this holds, so b is in $\mathrm{Col}(A)$ if and only if

$$b_2 + b_3 - 3b_1 = 0.$$

□

PROBLEM 5.2

For each of the following collections of vectors, decide whether it is a basis of $\mathbb{R}^3$. Explain your answer.

1. $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

SOLUTION

We row reduce the matrix whose columns are v_1, v_2, v_3.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + 2R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column has a pivot, so v_1, v_2, v_3 are linearly independent, and every row has a

SOLUTION (CONTINUED)

pivot, so they span $\mathbb{R}^3$. This collection **is** a basis of $\mathbb{R}^3$. □

2. $v_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $v_4 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

SOLUTION

This collection is **not** a basis of $\mathbb{R}^3$. Four vectors in $\mathbb{R}^3$ are always linearly dependent: the 3×4 matrix with these columns has at most 3 pivots, so at least one column has no pivot and the homogeneous system has a nontrivial solution. In fact, by part (a) the vectors v_1, v_2, v_3 already span $\mathbb{R}^3$, so v_4 is a linear combination of them. □

3. $v_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

SOLUTION

This collection is **not** a basis of $\mathbb{R}^3$. Two vectors cannot span $\mathbb{R}^3$: the 3×2 matrix with these columns has at most 2 pivots, so some row has no pivot, and the system $Ax = b$ is inconsistent for some b in $\mathbb{R}^3$. Geometrically, $\text{Span}\{v_1, v_2\}$ is a plane through the origin, and vectors off that plane are not in the span. □

4. $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$

SOLUTION

We row reduce the matrix whose columns are v_1, v_2, v_3.

$$\begin{bmatrix}1&0&1\\1&1&2\\0&1&1\end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix}1&0&1\\0&1&1\\0&1&1\end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$$

The third column has no pivot, so the vectors are linearly dependent. The reduced matrix

SOLUTION (CONTINUED)

says $v_3 = v_1 + v_2$, which we can also see directly. This collection is **not** a basis of $\mathbb{R}^3$. □

PROBLEM 5.3

The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent.

1. Explain why v_1, v_2 is not a basis of $\mathbb{R}^4$.

SOLUTION

Every basis of $\mathbb{R}^4$ has exactly 4 vectors, since $\dim \mathbb{R}^4 = 4$. Concretely, the 4×2 matrix with columns v_1, v_2 has at most 2 pivots, so two of its rows have no pivot, and $x_1 v_1 + x_2 v_2 = b$ is inconsistent for some b in $\mathbb{R}^4$. So v_1, v_2 do not span $\mathbb{R}^4$. □

2. Find a basis of $\mathbb{R}^4$ that includes v_1 and v_2.

SOLUTION

We need two more vectors, and there are many choices. Let's try adding

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and row reduce the matrix whose columns are v_1, v_2, v_3, v_4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Every column has a pivot, so v_1, v_2, v_3, v_4 are linearly independent, and every row has a pivot, so they span $\mathbb{R}^4$. So v_1, v_2, v_3, v_4 is a basis of $\mathbb{R}^4$ that includes v_1 and v_2. □

PROBLEM 5.4

Let $V = \text{Span}\{v_1, v_2\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

and let $\mathcal{B} = \{v_1, v_2\}$.

1. Explain why $\mathcal{B}$ is a basis of V.

SOLUTION

The vectors v_1, v_2 span V by the definition of V. They are linearly independent because neither is a multiple of the other. A linearly independent collection that spans V is a basis of V, so $\mathcal{B}$ is a basis of V, and $\dim V = 2$. □

2. The vector $w = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ is in V. Find $[w]_{\mathcal{B}}$.

SOLUTION

We need to solve $c_1 v_1 + c_2 v_2 = w$, which is a system of three equations in the two unknowns c_1, c_2.

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 0 & 2 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2} R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

So $c_1 = 3$ and $c_2 = -1$, that is,

$$[w]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

As a check, $3v_1 - v_2 = (3 - 1, 3 + 1, 0 - 2) = (2, 4, -2)$. □

3. Is the vector $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in V? If it is, find $[u]_{\mathcal{B}}$.

SOLUTION

We try to solve $c_1v_1 + c_2v_2 = u$.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 2 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

The last row is the equation $0 = 4$, so the system is inconsistent. There is no way to write u as a linear combination of v_1 and v_2, so u is **not** in V, and it has no $\mathcal{B}$-coordinates. □