MATH 221 · FALL 2026

Problem Set 6: Rank, Nullity, and Bases

DUE Friday, October 16th by 11:59pm

> *First, there's the room you can see through the glass—that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair—all but the bit behind the fireplace. Oh! I do so wish I could see* that *bit!*
>
> From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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PROBLEM 6.1

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

You may use the Row Reducer to help you calculate.

1. Find $\text{rank}(A)$ and $\text{nullity}(A)$.
2. Is the equation $Ax = b$ consistent for every vector b in $\mathbb{R}^4$? Explain.
3. Let

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

   Explain why w_1, w_2 is a basis of $\text{Nul}(A)$.

PROBLEM 6.2

1. A is a 3×7 matrix, and the solution set of $Ax = 0$ is a plane through the origin. What is $\text{rank}(A)$?

2. A is a 6×4 matrix, and for some vector b the equation $Ax = b$ has exactly one solution. What is $\text{rank}(A)$?

3. A is a 3×5 matrix, and $\text{Col}(A)$ is a line through the origin. How many free variables does the equation $Ax = 0$ have?

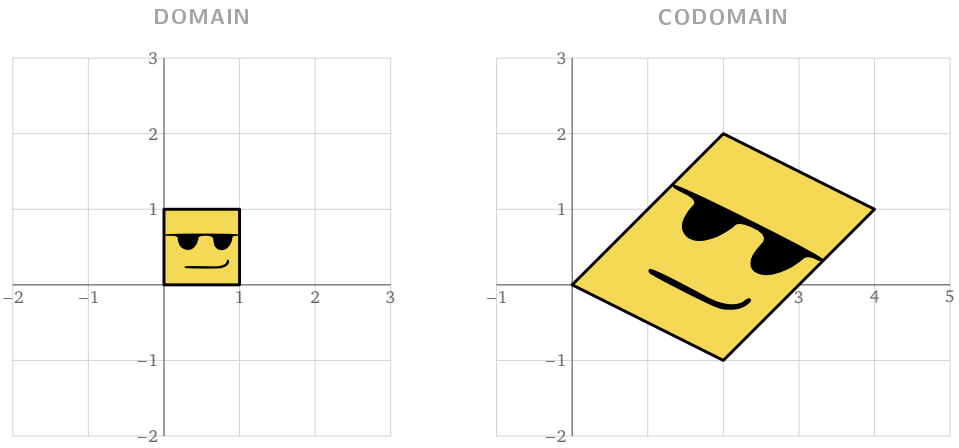
4. A is a 3×5 matrix, explain why the solution set of $Ax = 0$ cannot be a line through the origin.

PROBLEM 6.3

In this problem we will play a game called **Name That Matrix!** In each part you are given a description of a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, and your job is to find the matrix A.

1. Find the matrix A such that $T_A(e_1) = (3, 4)$ and $T_A(e_2) = (-1, 0)$.

2. Find the matrix A such that T_A maps the face on the left to the face on the right.

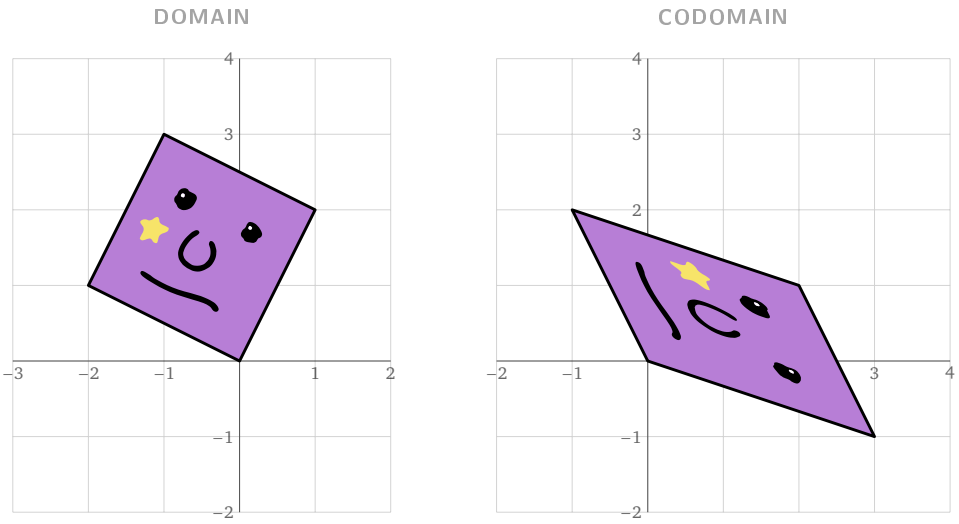


3. Find the matrix A such that T_A doubles the length of every vector and rotates it 180° around the origin.

4. Find the matrix A such that $T_A(1, 1) = (8, 1)$ and $T_A(1, -1) = (2, -3)$.

5. Find the matrix A such that T_A reflects the plane across the line consisting of the points $(3t, 4t)$.

6. Find the matrix A such that T_A maps the face on the left to the face on the right.



PROBLEM 6.4

Let P be the plane in $\mathbb{R}^3$ with equation $x + y + z = 0$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

lie in P, and the vector $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is perpendicular to P. Let $\mathcal{B} = \{v_1, v_2, w\}$.

1. Explain why $\mathcal{B}$ is a basis of $\mathbb{R}^3$.
2. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[u]_{\mathcal{B}}$.
3. Let u' be the reflection of u across the plane P. Note that reflecting in P leaves every vector in P where it is and sends w to $-w$. Find $[u']_{\mathcal{B}}$.
4. Find u'.
5. Let L be the line through the origin in the direction of w. Rotating $\mathbb{R}^3$ halfway around L sends w to itself and sends every vector in P to its negative. Find the vector z obtained by rotating u halfway around L.