MATH 221 · FALL 2026

Problem Set 6: Rank, Nullity, and Bases

SOLUTIONS

> *First, there's the room you can see through the glass—that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair—all but the bit behind the fireplace. Oh! I do so wish I could see* that *bit!*
>
> From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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PROBLEM 6.1

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

You may use the Row Reducer to help you calculate.

1. Find $\text{rank}(A)$ and $\text{nullity}(A)$.

   SOLUTION

   We row reduce A to row echelon form.

SOLUTION (CONTINUED)

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are pivots in columns 1, 3, and 5, so $\operatorname{rank}(A) = 3$. Columns 2 and 4 have no pivot, so x_2 and x_4 are free variables and $\operatorname{nullity}(A) = 2$. As a check, $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 2 = 5$, the number of columns of A. □

2. Is the equation $Ax = b$ consistent for every vector b in $\mathbb{R}^4$? Explain.

SOLUTION

No. The equation $Ax = b$ is consistent exactly when b is in $\operatorname{Col}(A)$, and $\dim \operatorname{Col}(A) = \operatorname{rank}(A) = 3$. Since $\dim \mathbb{R}^4 = 4$, the column space is not all of $\mathbb{R}^4$, so there are vectors b in $\mathbb{R}^4$ for which $Ax = b$ has no solution. We can also see this from the echelon form: the last row of the echelon form of A is zero, so for a suitable b the augmented matrix reduces to one with a row $[\,0\ 0\ 0\ 0\ 0 \mid c\,]$ with $c \neq 0$. □

3. Let

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Explain why w_1, w_2 is a basis of $\operatorname{Nul}(A)$.

SOLUTION

We multiply:

$$Aw_1 = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0-1+0 \\ 2+0+1-3+0 \\ 1+0+1-2+0 \\ 0+0+1-1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$Aw_2 = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+0-1+0 \\ -2+4+1-3+0 \\ -1+2+1-2+0 \\ 0+0+1-1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So both w_1 and w_2 are in $\text{Nul}(A)$. The vectors w_1 and w_2 are linearly independent, since neither is a multiple of the other (look at the first two entries). By part (a), $\dim \text{Nul}(A) = \text{nullity}(A) = 2$. Two linearly independent vectors in a 2-dimensional subspace automatically span it, so w_1, w_2 is a basis of $\text{Nul}(A)$. □

PROBLEM 6.2

1. A is a 3×7 matrix, and the solution set of $Ax = 0$ is a plane through the origin. What is $\text{rank}(A)$?

SOLUTION

The solution set of $Ax = 0$ is $\text{Nul}(A)$, and a plane through the origin is a 2-dimensional subspace, so $\text{nullity}(A) = 2$. By the Rank-Nullity Theorem, $\text{rank}(A) + \text{nullity}(A) = 7$, so $\text{rank}(A) = 5$. □

2. A is a 6×4 matrix, and for some vector b the equation $Ax = b$ has exactly one solution. What is $\text{rank}(A)$?

SOLUTION

If v_0 is a solution of $Ax = b$, then the solutions of $Ax = b$ are $x = v_0 + \text{Nul}(A)$. Since there is only one solution, $\text{Nul}(A) = \{0\}$ and $\text{nullity}(A) = 0$. The Rank-Nullity Theorem implies $\text{rank}(A) = 4 - 0 = 4$. □

3. A is a 3×5 matrix, and $\text{Col}(A)$ is a line through the origin. How many free variables does the equation $Ax = 0$ have?

SOLUTION

A line through the origin is a 1-dimensional subspace, so $\text{rank}(A) = \dim \text{Col}(A) = 1$. By the Rank-Nullity Theorem, $\text{nullity}(A) = 5 - 1 = 4$. The nullity is the number of free variables, so $Ax = 0$ has 4 free variables. □

4. A is a 3×5 matrix, explain why the solution set of $Ax = 0$ cannot be a line through the origin.

SOLUTION

If the solution set of $Ax = 0$ were a line through the origin, then $\text{nullity}(A) = 1$ and the Rank-Nullity Theorem would give $\text{rank}(A) = 5 - 1 = 4$. But the rank is the number of pivots, and a matrix with 3 rows has at most 3 pivots, since each row contains at most one pivot. So no such matrix exists. □

PROBLEM 6.3

In this problem we will play a game called **Name That Matrix!** In each part you are given a description of a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, and your job is to find the matrix A.

1. Find the matrix A such that $T_A(e_1) = (3, 4)$ and $T_A(e_2) = (-1, 0)$.

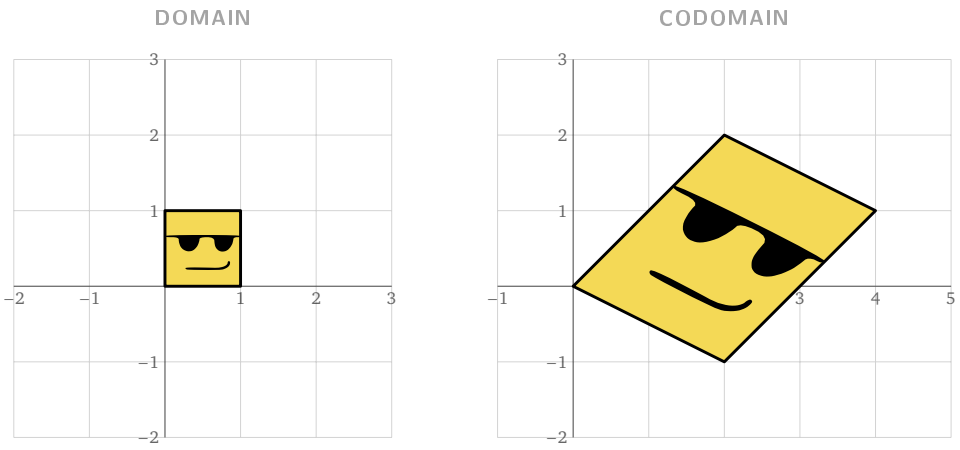
SOLUTION

The columns of A are $T_A(e_1)$ and $T_A(e_2)$, so

$$A = \begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix}.$$

□

2. Find the matrix A such that T_A maps the face on the left to the face on the right.



SOLUTION

The face on the left is the unit square, with corners at the origin, e_1, e_2, and $e_1 + e_2$. The mouth runs along the bottom edge, from the origin to e_1, and the face on the right has its mouth along the edge from the origin to $(2, -1)$. So $T_A(e_1) = (2, -1)$, and the other edge at the origin tells us $T_A(e_2) = (2, 2)$. These are the columns of A:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}.$$

As a check, the remaining corner $e_1 + e_2$ is sent to $(2, -1) + (2, 2) = (4, 1)$, which matches the picture. □

3. Find the matrix A such that T_A doubles the length of every vector and rotates it 180° around the origin.

SOLUTION

Rotating a vector v by 180° around the origin gives $-v$, and doubling its length gives $-2v$. So $T_A(e_1) = (-2, 0)$ and $T_A(e_2) = (0, -2)$, and

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$ □

4. Find the matrix A such that $T_A(1, 1) = (8, 1)$ and $T_A(1, -1) = (2, -3)$.

SOLUTION

We need $T_A(e_1)$ and $T_A(e_2)$, so we write e_1 and e_2 in terms of the vectors we know about:

$$e_1 = \tfrac{1}{2}(1, 1) + \tfrac{1}{2}(1, -1), \qquad e_2 = \tfrac{1}{2}(1, 1) - \tfrac{1}{2}(1, -1).$$

Since T_A is linear,

$$\begin{aligned} T_A(e_1) &= \tfrac{1}{2}T_A(1, 1) + \tfrac{1}{2}T_A(1, -1) = \tfrac{1}{2}(8, 1) + \tfrac{1}{2}(2, -3) = (5, -1), \\ T_A(e_2) &= \tfrac{1}{2}T_A(1, 1) - \tfrac{1}{2}T_A(1, -1) = \tfrac{1}{2}(8, 1) - \tfrac{1}{2}(2, -3) = (3, 2). \end{aligned}$$

So

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix}.$$

As a check, $A(1, 1) = (5 + 3, -1 + 2) = (8, 1)$ and $A(1, -1) = (5 - 3, -1 - 2) = (2, -3)$. □

5. Find the matrix A such that T_A reflects the plane across the line consisting of the points $(3t, 4t)$.

SOLUTION

Let $v = (3, 4)$, which lies on the line, and let $w = (-4, 3)$, which is perpendicular to the line. Reflecting across the line fixes v and reverses w, so $T_A(v) = v$ and $T_A(w) = -w$. Solving $c_1v + c_2w = e_1$ and $c_1v + c_2w = e_2$ gives

$$e_1 = \tfrac{3}{25}v - \tfrac{4}{25}w, \qquad e_2 = \tfrac{4}{25}v + \tfrac{3}{25}w.$$

Since T_A is linear,

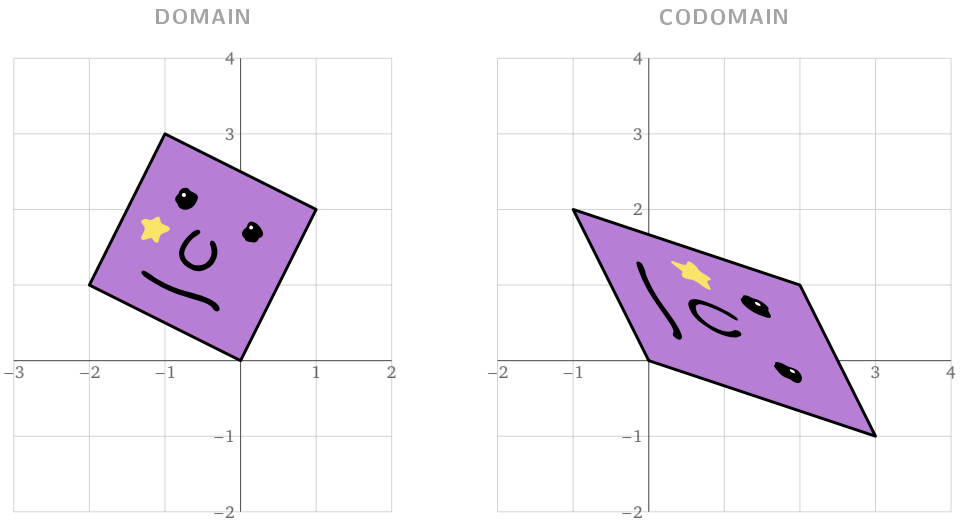
$$T_A(e_1) = \tfrac{3}{25}T_A(v) - \tfrac{4}{25}T_A(w) = \tfrac{3}{25}v + \tfrac{4}{25}w = \tfrac{1}{25}(9 - 16, 12 + 12) = \tfrac{1}{25}(-7, 24),$$
$$T_A(e_2) = \tfrac{4}{25}T_A(v) + \tfrac{3}{25}T_A(w) = \tfrac{4}{25}v - \tfrac{3}{25}w = \tfrac{1}{25}(12 + 12, 16 - 9) = \tfrac{1}{25}(24, 7).$$

So

$$A = \frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}.$$

As a check, $A(3, 4) = \frac{1}{25}(-21+96, 72+28) = (3, 4)$ and $A(-4, 3) = \frac{1}{25}(28+72, -96+21) = (4, -3) = -w$. □

6. Find the matrix A such that T_A maps the face on the left to the face on the right.



SOLUTION

This time the face on the left is not the unit square. Its corners are the origin, $u_1 = (-2, 1)$, $u_2 = (1, 2)$, and $u_1 + u_2 = (-1, 3)$, with the mouth along the edge from the origin to u_1. On the right, the mouth runs along the edge from the origin to $(-1, 2)$, and the other edge at the origin ends at $(3, -1)$. So $T_A(u_1) = (-1, 2)$ and $T_A(u_2) = (3, -1)$.
Since u_1, u_2 is a basis of $\mathbb{R}^2$, we can write e_1 and e_2 in terms of them. Solving $c_1u_1+c_2u_2 = e_1$

SOLUTION (CONTINUED)

and $c_1u_1 + c_2u_2 = e_2$ gives

$$e_1 = -\tfrac{2}{5}u_1 + \tfrac{1}{5}u_2, \qquad e_2 = \tfrac{1}{5}u_1 + \tfrac{2}{5}u_2.$$

Since T_A is linear,

$$T_A(e_1) = -\tfrac{2}{5}T_A(u_1) + \tfrac{1}{5}T_A(u_2) = -\tfrac{2}{5}(-1,2) + \tfrac{1}{5}(3,-1) = \tfrac{1}{5}(5,-5) = (1,-1),$$
$$T_A(e_2) = \tfrac{1}{5}T_A(u_1) + \tfrac{2}{5}T_A(u_2) = \tfrac{1}{5}(-1,2) + \tfrac{2}{5}(3,-1) = \tfrac{1}{5}(5,0) = (1,0).$$

So

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

As a check, $Au_1 = (-2+1, 2) = (-1, 2)$ and $Au_2 = (1+2, -1) = (3, -1)$. □

PROBLEM 6.4

Let P be the plane in $\mathbb{R}^3$ with equation $x + y + z = 0$. The vectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

lie in P, and the vector $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is perpendicular to P. Let $\mathcal{B} = \{v_1, v_2, w\}$.

1. Explain why $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

SOLUTION

We row reduce the matrix whose columns are v_1, v_2, w.

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Every column has a pivot, so v_1, v_2, w are linearly independent. Since $\dim \mathbb{R}^3 = 3$, any three linearly independent vectors in $\mathbb{R}^3$ automatically span $\mathbb{R}^3$. So $\mathcal{B}$ is a basis of $\mathbb{R}^3$. □

2. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[u]_{\mathcal{B}}$.

SOLUTION

We solve $c_1v_1 + c_2v_2 + c_3w = u$ by row reducing the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{3} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $c_1 = -1$, $c_2 = -1$, $c_3 = 2$, that is,

$$[u]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

As a check, $-v_1 - v_2 + 2w = (-1 - 0 + 2, 1 - 1 + 2, 0 + 1 + 2) = (1, 2, 3)$. □

3. Let u' be the reflection of u across the plane P. Note that reflecting in P leaves every vector in P where it is and sends w to $-w$. Find $[u']_{\mathcal{B}}$.

SOLUTION

By part (b), $u = -v_1 - v_2 + 2w$. Reflecting in P fixes v_1 and v_2 and reverses w, so

$$u' = -v_1 - v_2 - 2w, \qquad \text{that is,} \qquad [u']_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}. \qquad \square$$

4. Find u'.

SOLUTION

We convert back to standard coordinates:

$$u' = -v_1 - v_2 - 2w = -\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}.$$

As a check, u and u' should differ by a multiple of w, and indeed $u - u' = (4, 4, 4) = 4w$. □

5. Let L be the line through the origin in the direction of w. Rotating $\mathbb{R}^3$ halfway around L sends w to itself and sends every vector in P to its negative. Find the vector z obtained by rotating u halfway around L.

SOLUTION

This time the v_1 and v_2 coordinates change sign while the w coordinate stays the same, so

$$z = v_1 + v_2 + 2w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Note that $z = -u'$: rotating halfway around L is the same as reflecting in P and then negating. □