MATH 221 · FALL 2026

Problem Set 7: Injective, Surjective, and Invertible

DUE Friday, October 30th by 11:59pm

> *The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living.*
>
> From *Science and Method* by Henri Poincaré

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions.

PROBLEM 7.1

For each matrix A below, decide whether T_A is injective, surjective, and/or bijective. Justify your answers. You may use the Row Reducer to help you calculate.

1. $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$
4. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

PROBLEM 7.2

A tiny camera takes black-and-white photos that are a single row of four pixels, with brightness values x_1, x_2, x_3, x_4. The photographer's hands shake, so instead of the sharp image $x = (x_1, x_2, x_3, x_4)$ the

camera records a blurred image $b = (b_1, b_2, b_3)$ in which each pixel is the average of two neighboring sharp pixels:

$$b_1 = \tfrac{1}{2}(x_1 + x_2), \qquad b_2 = \tfrac{1}{2}(x_2 + x_3), \qquad b_3 = \tfrac{1}{2}(x_3 + x_4).$$

In other words, $b = Bx$ where

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1. Explain why T_B is not injective by finding a sharp image other than $(4, 4, 4, 4)$ that blurs to $b = (4, 4, 4)$.

2. Is every vector b in $\mathbb{R}^3$ the blurred image of some sharp image? Explain.

3. The camera is upgraded so that it also records the first pixel x_1 sharply. The recorded data is now $y = (x_1, b_1, b_2, b_3) = Cx$, where

$$C = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

   Explain why C is invertible.

4. Find C^{-1}.

5. The upgraded camera records $y = (2, 3, 4, 5)$. Recover the sharp image.

PROBLEM 7.3

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example of a matrix A for which it fails.

1. If A is a 3×5 matrix, then T_A is not injective.

2. If A is a 3×5 matrix and T_A is surjective, then the solution set of $Ax = 0$ is a plane through the origin.

3. If A is a 2×3 matrix, then T_A is surjective.

4. If A is a 2×2 matrix with $A^{-1} = A$, then $A = I_2$ or $A = -I_2$.

5. If A is a 4×4 matrix and there are two different vectors $x \neq y$ in $\mathbb{R}^4$ with $Ax = Ay$, then T_A is not surjective.

6. If the columns of A are linearly independent, then A is invertible.

PROBLEM 7.4

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

You may use the Row Reducer to help you calculate.

1. Find A^{-1}.
2. Use A^{-1} to solve $Ax = b$ for

$$b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \text{and for} \quad b = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}.$$

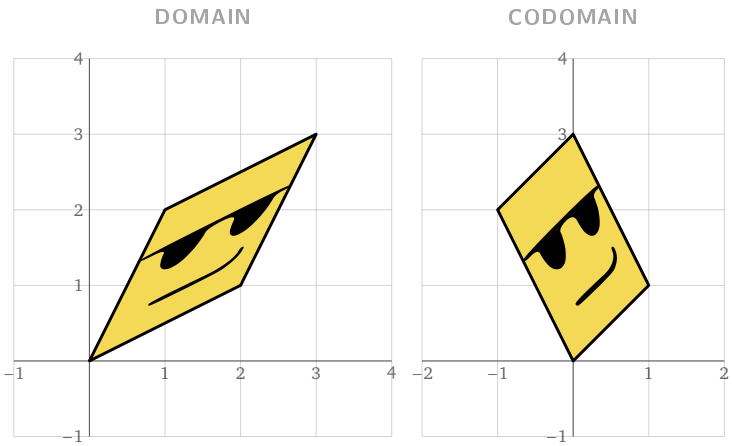
3. Explain why the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

is not invertible.

PROBLEM 7.5

The face on the left sits on the parallelogram with corners at the origin, $u_1 = (2, 1)$, $u_2 = (1, 2)$, and $u_1 + u_2 = (3, 3)$. We want to find the matrix A such that T_A maps the face on the left to the face on the right. Let $w_1 = T_A(u_1)$ and $w_2 = T_A(u_2)$.



1. Define $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$ be the 2×2 matrices with these vectors as columns. Explain why $AU = W$.
2. Use the equation $AU = W$ to calculate A.
   Hint: the 2×2 inverse formula.