MATH 221 · FALL 2026

Problem Set 7: Injective, Surjective, and Invertible

SOLUTIONS

The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living.

From *Science and Method* by Henri Poincaré

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions.

PROBLEM 7.1

For each matrix A below, decide whether T_A is injective, surjective, and/or bijective. Justify your answers. You may use the Row Reducer to help you calculate.

1. $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$

SOLUTION

We row reduce A to row echelon form.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Every column has a pivot, so $Ax = 0$ has only the trivial solution and T_A is injective. The third row has no pivot, so $\operatorname{rank}(A) = 2 < 3$ and $\operatorname{Col}(A)$ is not all of $\mathbb{R}^3$. So T_A is not surjective, and therefore not bijective. □

2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

SOLUTION

We row reduce A to row echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \end{bmatrix}$$

Every row has a pivot, so $\text{rank}(A) = 2$, $\text{Col}(A) = \mathbb{R}^2$, and T_A is surjective. The third column has no pivot, so x_3 is a free variable and $Ax = 0$ has nonzero solutions. So T_A is not injective, and therefore not bijective. □

3. $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$

SOLUTION

We row reduce A to row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are only two pivots. The third column has no pivot, so $Ax = 0$ has nonzero solutions and T_A is not injective. The third row has no pivot, so $\text{rank}(A) = 2 < 3$ and T_A is not surjective. So T_A is neither injective nor surjective, and not bijective. □

4. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

SOLUTION

We row reduce A to row echelon form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

SOLUTION (CONTINUED)

There is a pivot in every column, so T_A is injective, and a pivot in every row, so T_A is surjective. So T_A is bijective, and A is invertible. □

PROBLEM 7.2

A tiny camera takes black-and-white photos that are a single row of four pixels, with brightness values x_1, x_2, x_3, x_4. The photographer's hands shake, so instead of the sharp image $x = (x_1, x_2, x_3, x_4)$ the camera records a blurred image $b = (b_1, b_2, b_3)$ in which each pixel is the average of two neighboring sharp pixels:

$$b_1 = \tfrac{1}{2}(x_1 + x_2), \qquad b_2 = \tfrac{1}{2}(x_2 + x_3), \qquad b_3 = \tfrac{1}{2}(x_3 + x_4).$$

In other words, $b = Bx$ where

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1. Explain why T_B is not injective by finding a sharp image other than $(4, 4, 4, 4)$ that blurs to $b = (4, 4, 4)$.

SOLUTION

The equation $Bx = (4, 4, 4)$ says

$$x_1 + x_2 = 8, \qquad x_2 + x_3 = 8, \qquad x_3 + x_4 = 8.$$

Subtracting the second equation from the first gives $x_1 = x_3$, and subtracting the third from the second gives $x_2 = x_4$. So the solutions are the images $(s, 8 - s, s, 8 - s)$, and the solution set is

$$x = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

For example, the sharp image $(5, 3, 5, 3)$ also blurs to $(4, 4, 4)$. Two different sharp images have the same blurred image, so T_B is not injective. No unblurring procedure can exist, since it would have to send $(4, 4, 4)$ back to both $(4, 4, 4, 4)$ and $(5, 3, 5, 3)$. □

2. Is every vector b in $\mathbb{R}^3$ the blurred image of some sharp image? Explain.

SOLUTION

Yes. The matrix B is already in row echelon form, with a pivot in each of its three rows. So $\text{rank}(B) = 3$, $\text{Col}(B) = \mathbb{R}^3$, and T_B is surjective. Every b in $\mathbb{R}^3$ is the blurred image of

SOLUTION (CONTINUED)

some sharp image, and by part (a) in fact of infinitely many. □

3. The camera is upgraded so that it also records the first pixel x_1 sharply. The recorded data is now $y = (x_1, b_1, b_2, b_3) = Cx$, where

$$C = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Explain why C is invertible.

SOLUTION

We row reduce C to row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

There is a pivot in every row and every column, so T_C is both surjective and injective. A square matrix whose transformation is bijective is invertible, so C is invertible. □

4. Find C^{-1}.

SOLUTION

The matrix C^{-1} sends the recorded data $y = Cx$ back to the sharp image x, so we solve the equations

$$y_1 = x_1, \qquad y_2 = \tfrac{1}{2}(x_1 + x_2), \qquad y_3 = \tfrac{1}{2}(x_2 + x_3), \qquad y_4 = \tfrac{1}{2}(x_3 + x_4)$$

for the x_i in terms of the y_i, one pixel at a time:

$$\begin{aligned} x_1 &= y_1, \\ x_2 &= 2y_2 - x_1 = -y_1 + 2y_2, \\ x_3 &= 2y_3 - x_2 = y_1 - 2y_2 + 2y_3, \\ x_4 &= 2y_4 - x_3 = -y_1 + 2y_2 - 2y_3 + 2y_4. \end{aligned}$$

SOLUTION (CONTINUED)

Reading off the coefficients,

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 2 \end{bmatrix}.$$

Row reducing $[\, C \mid I_4 \,]$ gives the same answer. □

5. The upgraded camera records $y = (2, 3, 4, 5)$. Recover the sharp image.

SOLUTION

The sharp image is

$$x = C^{-1}y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2+6 \\ 2-6+8 \\ -2+6-8+10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}.$$

As a check, the first pixel is 2 and the averages of neighboring pixels are 3, 4, and 5, as recorded. □

PROBLEM 7.3

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example of a matrix A for which it fails.

1. If A is a 3×5 matrix, then T_A is not injective.

SOLUTION

True. A row echelon form of A has at most 3 pivots, one per row, but A has 5 columns. So at least 2 columns have no pivot, which means $Ax = 0$ has free variables and therefore nonzero solutions. So T_A is not injective. □

2. If A is a 3×5 matrix and T_A is surjective, then the solution set of $Ax = 0$ is a plane through the origin.

SOLUTION

True. Since T_A is surjective, $\text{Col}(A) = \mathbb{R}^3$ and $\text{rank}(A) = 3$. By the Rank-Nullity Theorem,

SOLUTION (CONTINUED)

$\text{nullity}(A) = 5 - 3 = 2$. So the solution set of $Ax = 0$ is the 2-dimensional subspace $\text{Nul}(A)$ of $\mathbb{R}^5$, which is a plane through the origin. □

3. If A is a 2×3 matrix, then T_A is surjective.

SOLUTION

False. For example, if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then $\text{rank}(A) = 1$, so $\text{Col}(A)$ is the line spanned by $(1, 1)$ rather than all of $\mathbb{R}^2$. The vector $(1, 0)$ is not in the image of T_A. □

4. If A is a 2×2 matrix with $A^{-1} = A$, then $A = I_2$ or $A = -I_2$.

SOLUTION

False. The condition $A^{-1} = A$ says that $AA = I_2$. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so $A^{-1} = A$, but A is neither I nor $-I$. Geometrically, T_A is the reflection across the line $y = x$, and reflecting twice returns every point to where it started. □

5. If A is a 4×4 matrix and there are two different vectors $x \neq y$ in $\mathbb{R}^4$ with $Ax = Ay$, then T_A is not surjective.

SOLUTION

True. Since $Ax = Ay$, we have $A(x - y) = 0$, and $x - y \neq 0$. So $\text{Nul}(A)$ contains a nonzero vector and $\text{nullity}(A) \geq 1$. By the Rank-Nullity Theorem, $\text{rank}(A) = 4 - \text{nullity}(A) \leq 3$, so $\text{Col}(A)$ is not all of $\mathbb{R}^4$ and T_A is not surjective. □

6. If the columns of A are linearly independent, then A is invertible.

SOLUTION

False. Only square matrices can be invertible. For example, the columns of the 3×2 matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

from Problem 1(a) are linearly independent, but T_A is not surjective, so T_A is not bijective and A is not invertible. □

PROBLEM 7.4

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

You may use the Row Reducer to help you calculate.

1. Find A^{-1}.

SOLUTION

We row reduce $[\, A \mid I \,]$ to reduced row echelon form.

SOLUTION (CONTINUED)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

The left half is now the identity, so the right half is the inverse:

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

As a check,

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3-3+1 & -3+5-2 & 1-2+1 \\ 3-6+3 & -3+10-6 & 1-4+3 \\ 3-9+6 & -3+15-12 & 1-6+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

2. Use A^{-1} to solve $Ax = b$ for

$$b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \qquad \text{and for} \qquad b = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}.$$

SOLUTION

Since A is invertible, $Ax = b$ has the unique solution $x = A^{-1}b$. For the first vector,

$$x = A^{-1}b = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6-9+5 \\ -6+15-10 \\ 2-6+5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

and for the second,

$$x = A^{-1}b = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 9-18+10 \\ -9+30-20 \\ 3-12+10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

As a check, the second answer says that the sum of the columns of A is $(3, 6, 10)$, which is true. □

3. Explain why the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

is not invertible.

SOLUTION

We row reduce $[\,B \mid I\,]$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 3 & 7 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \end{array}\right]$$

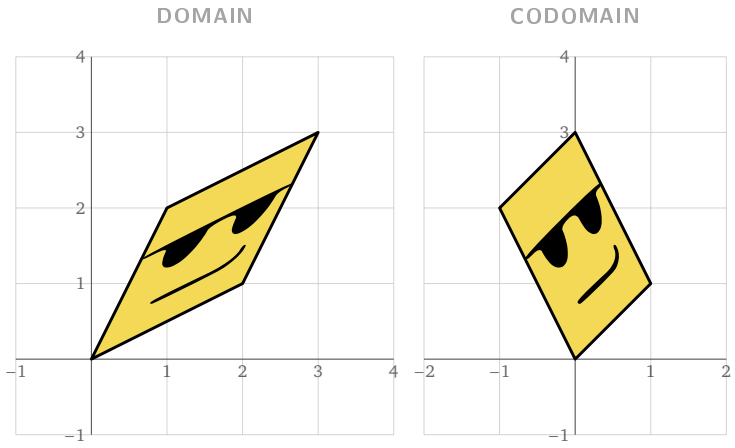
$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

The left half has only two pivots, so no further row operations can turn it into the identity matrix, and the algorithm for computing B^{-1} fails. Put another way, $\operatorname{rank}(B) = 2 < 3$, so T_B is not surjective. Since B is invertible exactly when T_B is bijective, B is not invertible. □

PROBLEM 7.5

The face on the left sits on the parallelogram with corners at the origin, $u_1 = (2, 1)$, $u_2 = (1, 2)$, and $u_1 + u_2 = (3, 3)$. We want to find the matrix A such that T_A maps the face on the left to the face on

the right. Let $w_1 = T_A(u_1)$ and $w_2 = T_A(u_2)$.



1. Define $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & w_2 \end{bmatrix}$ be the 2×2 matrices with these vectors as columns. Explain why $AU = W$.

SOLUTION

The columns of a product AU are A times the columns of U, so

$$AU = \begin{bmatrix} Au_1 & Au_2 \end{bmatrix} = \begin{bmatrix} T_A(u_1) & T_A(u_2) \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = W. \qquad \square$$

2. Use the equation $AU = W$ to calculate A.
Hint: the 2×2 inverse formula.

SOLUTION

Multiplying both sides of $AU = W$ on the right by U^{-1} gives $A = AUU^{-1} = WU^{-1}$. So

$$A = WU^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2+1 & -1-2 \\ 2-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

As a check, $A(2,1) = (1,1)$ and $A(1,2) = (-1,2)$, and the top corner goes to $A(3,3) = (0,3)$, as in the picture. $\square$