MATH 221 · FALL 2026

Problem Set 8: Rotations and Determinants

DUE Friday, November 6th by 11:59pm

And am I under some spell?
And do my thoughts belong to me?
Or just some slogan I ingested to save time?

From *Content Nausea* by Parquet Courts

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

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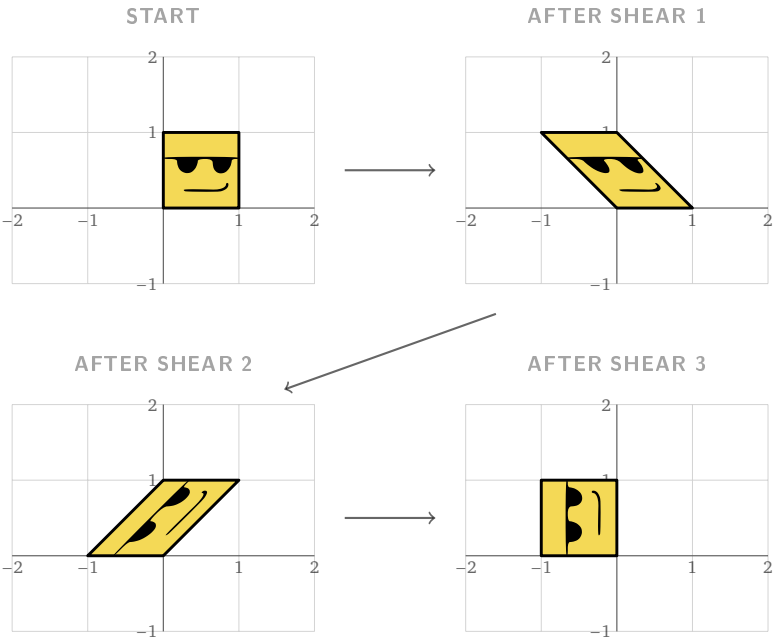
PROBLEM 8.1

A *shear* is a matrix of the form

$$H_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad \text{or} \qquad V_b = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}.$$

The horizontal shear H_a sends (x, y) to $(x + ay, y)$, so it slides the horizontal row at height y sideways by ay. The vertical shear V_b sends (x, y) to $(x, bx + y)$, so it slides the vertical column at position x up by bx. I encourage you to experiment with these to see how they work using the visualizer.

Digital images are grids of pixels. Shears are very easy for a computer to apply to a digital image, since they only slide pixels horizontally or vertically. To apply H_a, the computer slides each row of pixels sideways as a block, and the pixels within a row stay in the same order. Rotating an image directly is much harder, since most pixels land between the grid points and the computer has to blend neighboring pixels together. For this reason, image editing programs often rotate an image by applying three shears in a row. The pictures below show how three shears rotate the face counterclockwise by 90°.



1. Use the pictures to write the rotation matrix

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

as a product of three shears.

2. In fact, every rotation other than the rotation by 180° is a product of three shears. The counterclockwise rotation by an angle θ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let θ be the angle with $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$, so that

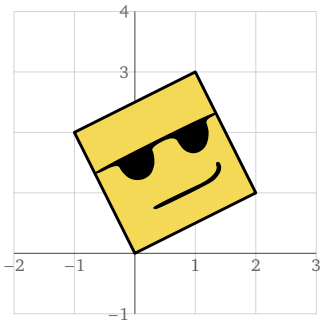
$$R_\theta = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Find numbers a, b, and c such that $R_\theta = H_a V_b H_c$.

3. (Challenge) Let θ be an angle that is not a multiple of 180°. Find numbers a, b, and c, depending on θ, such that $R_\theta = H_a V_b H_c$.

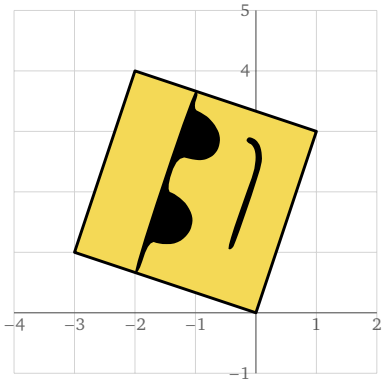
PROBLEM 8.2

The face below has been rotated a bit, so its sides are no longer parallel to the axes.

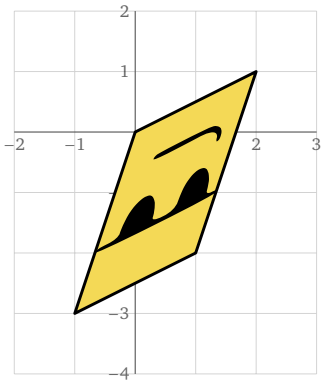


In each part, A is a 2×2 matrix and T_A maps the face above to the face shown. Use the picture to find $\det(A)$.

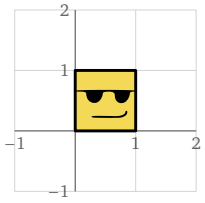
1.



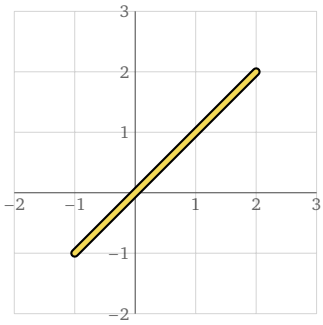
2.



3.



4. Sadly, in this part the face has been squished onto a line segment.



PROBLEM 8.3

Use row operations to answer the following questions. You may use the Row Reducer to help you calculate.

1. Find $\det(A)$, where
$$A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix}.$$

2. Find all values of c for which the matrix
$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & c \\ 1 & 3 & 4 \end{bmatrix}$$
is not invertible.

PROBLEM 8.4

Suppose that
$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5.$$
Find the determinant of each matrix below.

1. $\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$

2. $\begin{bmatrix} a & b & c \\ d+3a & e+3b & f+3c \\ g & h & i \end{bmatrix}$

3. $\begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$

4. $\begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a+4g & b+4h & c+4i \end{bmatrix}$

PROBLEM 8.5

Let A and B be 3×3 matrices with $\det(A) = 3$ and $\det(B) = -2$. Find each of the following.

1. $\det(AB)$
2. $\det(A^{-1})$
3. $\det(A^2B^{-1})$
4. $\det(A^{-1}BA)$
5. $\det(2A)$

PROBLEM 8.6

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example for which it fails.

1. If A and B are 2×2 matrices, then $\det(A + B) = \det(A) + \det(B)$.
2. If A is a square matrix with $A^2 = A$, then $\det(A) = 0$ or $\det(A) = 1$.
3. If A is a 4×4 matrix with $A^3 = 0$, then A is not invertible.
4. There is no 3×3 matrix A with $A^2 = -I_3$.
5. If A is a 2×2 matrix with $\det(A) = 1$, then T_A is a rotation.
6. If A is a 2×2 matrix with $A^6 = I_2$, then $\det(A) = 1$.