MATH 221 · FALL 2026

Problem Set 9: Eigenvalues and Eigenvectors

DUE Friday, November 20th by 11:59pm

Suddenly, without warning, his feverish activity was interrupted and was replaced by a kind of fascination. He spent several days as if he were bewitched, softly repeating to himself a string of fearful conjectures without giving credit to his own understanding. Finally, one Tuesday in December, at lunchtime, all at once he released the whole weight of his torment. The children would remember for the rest of their lives the august solemnity with which their father, devastated by his prolonged vigil and by the wrath of his imagination, revealed his discovery to them: 'The earth is round, like an orange.'

From *One Hundred Years of Solitude* by Gabriel García Márquez

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. You may consult the course notes, the supplementary textbook, your classmates, your professor, math interns, or Q-Center tutors for help with the assignment, but **all sources and collaborators must be explicitly referenced**. Everyone is expected to write their own solutions. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional and are for extra credit

---

PROBLEM 9.1

Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. For each eigenvalue of A, find a basis for its eigenspace.

PROBLEM 9.2

Let P be the plane in $\mathbb{R}^3$ with equation $x + y + z = 0$, and let L be the line spanned by

$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which is perpendicular to P. For each linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ below, find the eigenvalues of A and a basis for each eigenspace. **Hint:** You do not need to find the matrix A.

1. T_A is the reflection across P.
2. T_A is the projection onto P.
3. T_A is the rotation by 180° around the line L.
4. T_A is the rotation by 90° around the line L.

PROBLEM 9.3

A biologist is studying a population of songbirds, counting only the females. Each bird is a *chick* in its first year, a *yearling* in its second year, and an *adult* after that. The biologist observes the following.

- Each yearling and each adult raises an average of 3 chicks, which are the chicks of the following year.
- Half of the chicks survive to become yearlings the following year.
- Half of the yearlings survive to become adults the following year.
- Half of the adults survive to the following year.

Let c_k, y_k, and a_k be the numbers of chicks, yearlings, and adults in year k, and let

$$x_k = \begin{bmatrix} c_k \\ y_k \\ a_k \end{bmatrix}.$$

1. Find the matrix L such that $x_{k+1} = Lx_k$.
2. Find the eigenvalues of L.
3. The biologist wants to find a population of 900 birds whose proportions of chicks, yearlings, and adults stay the same from one year to the next. Find such a population.
4. Habitat loss lowers the fraction of chicks that survive their first year from $\frac{1}{2}$ to a number s, while everything else stays the same. For which value of s is there a population of birds that stays exactly the same from one year to the next?

PROBLEM 9.4

For each statement below, decide whether it is true or false. If it is true, explain why. If it is false, give an example for which it fails.

1. If A is a square matrix with $A^2 = A$, then every eigenvalue of A is 0 or 1.
2. A square matrix A is invertible if and only if 0 is not an eigenvalue of A.
3. If λ is an eigenvalue of a square matrix A, then λ^2 is an eigenvalue of A^2.
4. If v and w are eigenvectors of a square matrix A, then $v + w$ is an eigenvector of A.

5. If A is a square matrix, then A and A^T have the same eigenvalues.
6. If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A + B$.