MATH 301 · FALL 2026

Problem Set 1: Warm Up

SOLUTIONS

The journey of a thousand miles begins with one step.

From *Dao De Jing* by Laozi

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

PROBLEM 1.1

In this problem we review doing calculations in symmetric groups. **You do not need to show any work, just the final answers.**

1. Below are compositions of permutations expressed in cycle notation. Express each permutation as a collection of disjoint cycles; that is, calculate the cycle notation for the composite permutation.

   (a) $(134)(241)(1234)$

   (b) $(15)(14)(13)(12)$

   (c) $(15342)(43512)$

   SOLUTION

   (a) (24)

   (b) (12345)

   (c) ε □

2. Calculate the order of the following permutations. Beware, I may be trying to trick you!

   (a) $(123)(45)(67)$

   (b) $(325)(146)(27)$

   SOLUTION

   (a) Since the cycles are disjoint, the order is the least common multiple of the lengths,

SOLUTION (CONTINUED)

which is $\operatorname{lcm}(3,2,2) = 6$.

(b) First we express the permutation as a product of disjoint cycles,

$$(325)(146)(27) = (146)(2753).$$

Thus, the order of this permutation is $\operatorname{lcm}(3,4) = 12$. □

3. For each of the following permutations σ, determine the inverse permutation σ^{-1}.

   (a) (12345)
   (b) $(123)(456)$
   (c) $(12)(345)(6789)$

   Can you see a general recipe for finding the inverse of a permutation expressed in cycle notation?

SOLUTION

The general trick is to write each cycle in reverse order.

(a) (54321)

(b) $(321)(654)$

(c) $(21)(543)(9876)$ □

PROBLEM 1.2

1. Give an example of an abelian group that is not cyclic (and prove it).

SOLUTION

There are many examples, but the simplest is $G := C_2 \times C_2$. Every cyclic group is abelian, and direct products of abelian groups are abelian. Hence G is abelian. Every element in G has order dividing 2 while the group has order 4, hence G is not cyclic. □

2. Find two non-isomorphic groups of order 6 and prove they are not isomorphic.

SOLUTION

Let $G_1 = C_6$ be the cyclic group of order 6 and let $G_2 = S_3$ be the 3rd symmetric group.

SOLUTION (CONTINUED)

Both groups have order 6 but G_1 is abelian while G_2 is not. Thus $G_1 \ncong G_2$. □

3. What are the possible orders of elements in the symmetric group S_5? For each possible order, find an element in S_5 with that order.

SOLUTION

The possible orders are $1, 2, 3, 4, 5, 6$. Orders 1 through 5 may be achieved by cycles. Recall that every permutation σ may be expressed as a product of disjoint cycles $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ and that $\operatorname{ord}(\sigma) = \operatorname{lcm}\{\operatorname{ord}(\sigma_i) : 1 \le i \le k\}$. Cycles in S_5 can have any length between 1 and 5 and disjoint cycles must have lengths adding up to at most 5. Thus an example of an element of order 6 is $(12)(345)$. The only other type of non-cycle permutation is a product of two disjoint 2-cycles $(12)(34)$, but these have order 2. □

4. If $H \subseteq G$ is a subgroup with index 2, prove that H is a normal subgroup of G.

SOLUTION

The assumption that H has index 2 implies that for all $g \in G \setminus H$,

$$G = H \cup gH = H \cup Hg. \tag{1.0.1}$$

Since H is disjoint from gH and Hg, it then follows that $gH = Hg$ for all $g \in G \setminus H$. This identity also holds for all $g \in H$. Therefore $gHg^{-1} = H$ for all $g \in G$, which implies that H is a normal subgroup. □

5. Let D_6 be the 6th dihedral group (with order 12). Determine all the conjugacy classes of elements of D_6.

SOLUTION

Observe that

$$srs^{-1} = r^{-1}. \tag{1.0.2}$$

Hence $(r^j s)r^k(r^j s)^{-1} = r^{-k}$ for any k. Thus $\{r^k, r^{-k}\}$ is a conjugacy class for each $0 \le k < 6$. Two of these pairs degenerate to singletons,

$$\{1\}, \{r, r^5\}, \{r^2, r^4\}, \{r^3\}.$$

The relation (1.0.2) implies that $sr = r^{-1}s$. In other words, moving an s past an r inverts it. Thus

$$(r^j s)r^k s(r^j s)^{-1} = r^j(sr^k s)(sr^{-j}) = r^j(r^{-k}s^2)r^j s = r^{2j-k}s. \tag{1.0.3}$$

SOLUTION (CONTINUED)

Hence $r^k s \sim r^{2j-k}s$ for all $0 \leq j < 6$. This gives us two conjugacy classes,

$$\{s, r^2 s, r^4 s\} \text{ and } \{rs, r^3 s, r^5 s\}.$$

Conjugacy classes of elements and subgroups tend to have some qualitative description. For example, the group D_6 may be thought of as the group of linear symmetries of a regular hexagon. If we draw the hexagon inscribed in the unit circle with two vertices on the x-axis, then the conjugacy class $\{s, r^2 s, r^4 s\}$ corresponds to the reflections across vertex axes while the conjugacy class $\{rs, r^3 s, r^5 s\}$ corresponds to reflection across edge axes. □

PROBLEM 1.3

The table below gives the partial description of a mystery group G of order 8.

G	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	a	1	c	b	e	d	g	f
b	b	c	a	1	f	g	e	d
c	c	b	1	a	g	f	d	e
d	d	e	g	f	a	1	b	c
e	e	d	f	g	1	a	c	b
f	f	g	d	e	c	b	a	1
g	g	f	e	d	b	c	1	a

1. Using the axioms of a group and information provided, fill in the missing entries of the multiplication table for G. There is only one solution!
2. Is G isomorphic to D_4? Either find an isomorphism or prove that no such isomorphism exists.

SOLUTION

The group G is not isomorphic to D_4. Isomorphisms preserve the order of elements, hence isomorphic groups must have the same number of elements of order 4. The multiplication table for G implies that b, c, d, e, f, g all have order 4, whereas D_4 only has two elements of order 4 (namely rotation by 90 degrees in both directions.) Thus G and D_4 cannot be isomorphic. □

PROBLEM 1.4

Let G be a group and suppose that $H_1, H_2 \subseteq G$ are subgroups such that $H_i \neq G$.

1. Prove that $G \neq H_1 \cup H_2$.

SOLUTION

Suppose that $G = H_1 \cup H_2$. Since both H_i are proper, they must each contain an element $h_i \in H_i$ not contained in the other subgroup. But then $h_3 := h_1 h_2 \in G$, hence h_3 must belong to either H_1 or H_2. Without loss of generality suppose that $h_3 \in H_1$. Then $h_1^{-1} \in H_1$ hence $h_2 = h_1^{-1} h_3 \in H_1$, which contradicts our assumption that $h_2 \notin H_1$. This contradiction implies that $G \neq H_1 \cup H_2$. □

2. Find an example of a group G which can be expressed as a union of three proper subgroups $G = H_1 \cup H_2 \cup H_3$.

SOLUTION

Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and define

$$H_1 := \langle (1,0) \rangle$$
$$H_2 := \langle (0,1) \rangle$$
$$H_3 := \langle (1,1) \rangle.$$

Then each H_i has order 2, hence is proper, and $G = H_1 \cup H_2 \cup H_3$. □

3. (Challenge) Let G be a group. Prove that there are three proper subgroups $H_1, H_2, H_3 \subseteq G$ such that $G = H_1 \cup H_2 \cup H_3$ if and only if there is a surjective homomorphism $\rho : G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

SOLUTION

First, the easy direction. Suppose there exists a surjective homomorphism $\rho : G \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let

$$K_1 := \langle (1,0) \rangle$$
$$K_2 := \langle (0,1) \rangle$$
$$K_3 := \langle (1,1) \rangle,$$

and define $H_i := \rho^{-1}(K_i)$. Our assumption that ρ is surjective implies that each H_i is proper. Since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = K_1 \cup K_2 \cup K_3$, it follows that $G = H_1 \cup H_2 \cup H_3$. That completes the proof of the $\Leftarrow$ direction.
Now suppose that G has three proper subgroups $H_1, H_2, H_3 \subseteq G$ such that $G = H_1 \cup H_2 \cup H_3$. Throughout the rest of this solution, we will use i, j, k to denote three distinct elements of $\{1, 2, 3\}$. Let $N := H_1 \cap H_2 \cap H_3$. We prove several claims.

Claim 1: If $h_i \in H_i \setminus H_j$ and $h_j \in H_j \setminus H_i$, then $h_i h_j \in H_k$.

Proof. Let $g := h_i h_j$. Then $g \in G = H_1 \cup H_2 \cup H_3$. Suppose $g \in H_i$. Then $h_j = h_i^{-1} g \in H_i$, a contradiction. Hence $g \notin H_i$. A similar argument implies that $g \notin H_j$. Thus $g \in H_k$. □

SOLUTION (CONTINUED)

Claim 2: $H_i \cap H_j = N$.

Proof. Let $H := H_i \cap H_j$. Since $G \neq H_i \cup H_j$ by part 1, it follows that $H_k \not\subseteq H$. Suppose that $H \not\subseteq H_k$. Let $h \in H \setminus H_k$ and let $h_k \in H_k \setminus H$. Consider $g := hh_k \in G$. Then Claim 1 implies that $g \in H_1 \cap H_2 = H$. But then $h_k = h^{-1}g \in H$, a contradiction. Thus $H \subseteq H_k$. Hence

$$H \subseteq H \cap H_3 = H_1 \cap H_2 \cap H_3 = N.$$

On the other hand, $N \subseteq H$ by definition. Thus $H = N$. □

Claim 3: $[H_i : N] = 2$ for each i.

Proof. It suffices to show that for all $g_1, g_2 \in H_i \setminus N$ we have $g_1 g_2 \in N$. Suppose this were not the case; then $g_1 g_2 \in H_i \setminus N$. Problem 1.4.1 implies there is an element $h \in H_j \setminus N$. Then Claim 1 implies that $g_2 h$ and $g_1 g_2 h$ both belong to H_k. But then

$$g_1 = g_1(g_2h)(g_2h)^{-1} = (g_1g_2h)(g_2h)^{-1} \in H_k.$$

Hence $g_1 \in H_i \cap H_k = N$ by Claim 2. This contradicts our assumption that $g_1 \notin N$. Thus $g_1 g_2 \in N$. □

Claim 4: $[G : H_i] = 2$ for each i.

Proof. As in the previous claim, it suffices to prove that for all $g_1, g_2 \in G \setminus H_i$ we have $g_1 g_2 \in H_i$. There are essentially two cases: If $g_1, g_2 \in H_j \setminus N$, then Claim 3 implies that $g_1 g_2 \in N \subseteq H_i$. Otherwise $g_1 \in H_j \setminus N$ and $g_2 \in H_k \setminus N$. Then Claim 1 implies $g_1 g_2 \in H_i$. Thus $[G : H_i] = 2$. □

Now we put it altogether. Since each H_i has index 2, Problem 1.2.4 implies H_i is a normal subgroup. The intersection of normal subgroups is normal, thus N is a normal subgroup of G. Claims 3 and 4 together imply that

$$[G : N] = [G : H_i][H_i : N] = 2 \cdot 2 = 4.$$

Thus G/N is a group of order 4, hence must be isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Every $g \in G$ belongs to some H_i and Claim 3 implies that $g^2 \in N$. Hence every element of G/N has order dividing 2, which eliminates the cyclic case. Therefore $G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the composition of the natural quotient map and this isomorphism gives the desired surjection,

$$\rho : G \to G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ □

PROBLEM 1.5 · (Challenge)

Let p be an odd prime and suppose that G is a finite group of order $2p$. Prove that G is either cyclic or $G \cong D_p$, the pth dihedral group.

SOLUTION

Suppose G is not cyclic. Cauchy's theorem implies that G must contain elements r and s with orders p and 2 respectively. Note that $\langle r \rangle$ is a p-Sylow subgroup. Let n_p be the number of p-Sylow subgroups of G. Then Sylow's theorem implies that $n_p \mid 2$ and $n_p \equiv 1 \bmod p$, which leaves only one possibility: $n_p = 1$. Therefore $\langle r \rangle$ is a normal subgroup of G. Furthermore, srs also has order p, hence must be an element of $\langle r \rangle$. Thus $srs = r^i$, and s having order 2 implies that $i^2 \equiv 1 \bmod p$. As p is prime, the only solutions of this congruence are $i \equiv \pm 1 \bmod p$. If $i \equiv 1 \bmod p$, then $srs = r$, which implies that r and s commute. But in that case rs has order $2p$ which implies that G is cyclic, counter to our assumption. Thus $i = -1$ which implies that $srs = r^{-1}$. Therefore r and s satisfy the defining relations of the generators of D_p, giving us a surjective homomorphism $D_p \to G$. Both groups have order $2p$, hence this map must be an isomorphism. □