MATH 301 · FALL 2026

Problem Set 2: Group Actions

DUE Friday, September 11th at 11:59pm

"It seems very pretty," she said when she had finished it, "but it's rather hard to understand!" (You see she didn't like to confess, even to herself, that she couldn't make it out at all.) "Somehow it seems to fill my head with ideas—only I don't know exactly what they are!"

From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

Groups are an abstract, mathematical objectification of symmetry. Any group may be realized or *represented* as symmetries in many different ways. The point of *representation theory* is to study, classify, and relate these various *representations* of a group. In this course we primarily focus on two kinds of representations: *permutation* and *linear*. Permutation representations, also known as *group actions*, are the simpler of the two and serve as a proper appetizer before our main course.

2.1 · Group Actions

There are several equivalent ways of defining a permutation representation. We will opt for the simplest.

DEFINITION 2.1
Let G be a group and let X be a set. An **action** of G on X is a homomorphism $\rho : G \to S_X$. We call X, together with the homomorphism ρ, a **permutation representation** of G or, more briefly, a **G-set**.

An action of G on X is a way of turning elements of G into permutations of the set X. We will write ρ_g to mean $\rho(g)$. Hence $\rho_g \in S_X$ is the permutation that ρ associates to g. We often suppress the ρ when it is unambiguous in order to get a clean notation. For example, given $g \in G$ and $x \in X$ we write $g \cdot x$, $g(x)$, or simply gx as a shorthand for $\rho_g(x)$. Note that $gx \in X$. With this notation, the fact that ρ is a homomorphism gets hidden in an "associative law": if $g, h \in G$ and $x \in X$, then

$$(gh)x = g(hx).$$

Compare this with the literal, unpacked expression

$$\rho_{gh}(x) = (\rho_g \circ \rho_h)(x) = \rho_g(\rho_h(x)).$$

Thus we may simply write ghx without any parentheses.

DEFINITION 2.2
If X is a G-set, and $d = |X|$, then we say d is the **degree** of the representation.

DEFINITION 2.3
We say X is a **faithful** G-set if the homomorphism $\rho : G \to S_X$ is injective. In other words, X is faithful when $gx = x$ for all $x \in X$ implies $g = 1$.

Let's consider some examples.

EXAMPLE 2.4
Let $C_d = \langle \sigma \rangle$ be the cyclic group of order d and let $(12 \cdots d) \in S_d$ be a d-cycle. Then there is a unique homomorphism $\rho : C_d \to S_d$ such that $\rho(\sigma) = (12 \cdots d)$. This represents C_d as a particular d-cycle of the set $\{1, 2, \ldots, d\}$. This is a faithful representation of degree d.

PROBLEM 2.1

1. Can you find a degree 3 permutation representation of C_6?
2. Can you find a faithful degree 3 permutation representation of C_6? Find an example or prove it is impossible.

EXAMPLE 2.5
Let R be a commutative ring and let $\mathrm{Aff}(R)$ be the affine group of R. Given $\ell(x) = ax + b \in \mathrm{Aff}(R)$ and $r \in R$, the evaluation $\ell(r) = ar + b$ defines an action of $\mathrm{Aff}(R)$ on R. We call this the **natural** or **defining action** of $\mathrm{Aff}(R)$. The only detail that needs some thought is why evaluation *permutes* R. This is where it is important that the linear coefficient $a \in R^\times$ is a unit. In this example, our shorthand notation $\ell(r)$ feels quite natural. But keep in mind that the actual representation (which is hidden in the notation) turns the polynomial $\ell(x) = ax + b$ into a permutation of the (possibly infinite!) set R. Nevertheless, we do not need the full cycle description to calculate this permutation since $\ell(x)$ is given by a formula.

PROBLEM 2.2

Is the natural action of $\mathrm{Aff}(R)$ on R always faithful? Prove or find a counterexample.

EXAMPLE 2.6
Let S_d be the dth symmetric group. By construction, S_d acts on $\{1, 2, \ldots, d\}$. The homomorphism here is the identity $1 : S_d \to S_d$.

EXAMPLE 2.7

We can often extract permutation representations of a group G naturally from more complicated objects on which G acts as symmetries. For example, let $\mathcal{T}$ denote the rotational symmetry group of a regular tetrahedron. These are rotations of $\mathbb{R}^3$ which leave a regular tetrahedron appearing unchanged. Let V, E, F be the sets of vertices, edges, and faces of the tetrahedron. Observe that every rotational symmetry of of the tetrahedron must permute the elements of the sets V, E, F. Hence V, E, F are each naturally $\mathcal{T}$-sets with degrees 4, 6, and 4 respectively.

PROBLEM 2.3

1. Is V a faithful $\mathcal{T}$-set?
2. Label the vertices of the regular tetrahedron by 1, 2, 3, 4 and let $\rho_V : \mathcal{T} \to S_4$ be the vertices representation of $\mathcal{T}$. Is ρ_V surjective? If so, prove it; otherwise determine the image of ρ_V.

Does every group have a permutation representation? The answer is "Yes" for a silly reason. Note that S_1 is the trivial group with one element. Every group G has a unique homomorphism to S_1 which sends every element of G to the identity. This is called the **trivial representation**.

A more interesting question is whether every group G has a *faithful* permutation representation. Here the answer is again "Yes" but it takes more thought. The trick is to let G act on itself.

PROBLEM 2.4

Let G be a group. Let $\tau : G \to S_G$ be the function defined by $\tau_g(h) := gh$. We call τ the **left translation** map. Prove that τ is an injective homomorphism.

The result you proved in Problem 2.4 is known as Cayley's Theorem. This shows that *every* group may be embedded into a symmetric group. Hence no matter how abstract a group G is, no matter how weird the operation, Cayley's theorem implies we can always make G concrete by realizing it as a subgroup of some symmetric group. Hence the weird operation becomes composition of functions. This is a philosophically important result as it shows us that all abstract groups are, in a sense "real".

It is also practical: Say you wanted to implement groups in a computer. There is no way you will be able to dream up all the possible kinds of operations that group may have in advance. But Cayley's theorem tells us that you don't need to. We can just implement permutations and their composition, and then every group may be constructed in that framework.

However, on the flip-side, the left translation embedding of a group G into S_G can be incredibly inefficient. Note that $|S_G| = |G|!$ which is *way* bigger than G itself. So for large groups, this representation is impractical. Thus, for the sake of efficiency, it is often desirable to find a low-degree faithful permutation representation of G. For example, left-translation embeds S_d into S_{S_d}, which has order $d!!$. However, by construction S_d embeds into itself via the identity, which is a faithful degree d representation.

2.2 · Maps Between G-Sets

We have now defined our primary objects of interest: G-sets. When starting to build a new theory, the next step is to pin down the right kinds of maps between objects. The goal is to find maps which respect the essential structure of your objects. The general name for these kinds of structural maps is **morphism**. Along these lines, we also want to establish what it means for two G-sets to be *essentially the same* or **isomorphic**. If two G-sets X and Y are isomorphic, they should be in some sense fundamentally the same even if they are superficially different.

PROBLEM 2.5

Let G be a group and let X, Y be G-sets.

1. How should we define morphisms $f : X \to Y$? Explain why your notion is the right one.
2. Suppose X, Y, Z are G-sets and let $f_1 : Y \to Z$ and $f_2 : X \to Y$ be morphisms (following your definition). Prove that $f_1 f_2 : X \to Z$ is a morphism and prove that the identity function $1_X : X \to X$ defined by $1_X(x) = x$ is a morphism. (If you cannot prove this, then you should rethink your definition.)
3. How should we define what it means for two G-sets to be isomorphic? Explain why your notion is the right one.
4. Give an example of two G-sets which are isomorphic while not literally equal.
5. Find an example of a group G and two nontrivial G-sets X, Y for which there exists a morphism $f : X \to Y$ which is not an isomorphism.

Note, we sometimes refer to a morphism of G-sets as a **G-equivariant map**.