MATH 301 · FALL 2026

Problem Set 2: Group Actions

SOLUTIONS

"It seems very pretty," she said when she had finished it, "but it's rather hard to understand!" (You see she didn't like to confess, even to herself, that she couldn't make it out at all.) "Somehow it seems to fill my head with ideas—only I don't know exactly what they are!"

From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

Groups are an abstract, mathematical objectification of symmetry. Any group may be realized or *represented* as symmetries in many different ways. The point of *representation theory* is to study, classify, and relate these various *representations* of a group. In this course we primarily focus on two kinds of representations: *permutation* and *linear*. Permutation representations, also known as *group actions*, are the simpler of the two and serve as a proper appetizer before our main course.

2.1 · Group Actions

There are several equivalent ways of defining a permutation representation. We will opt for the simplest.

DEFINITION 2.1
Let G be a group and let X be a set. An **action** of G on X is a homomorphism $\rho : G \to S_X$. We call X, together with the homomorphism ρ, a **permutation representation** of G or, more briefly, a **G-set**.

An action of G on X is a way of turning elements of G into permutations of the set X. We will write ρ_g to mean $\rho(g)$. Hence $\rho_g \in S_X$ is the permutation that ρ associates to g. We often suppress the ρ when it is unambiguous in order to get a clean notation. For example, given $g \in G$ and $x \in X$ we write $g \cdot x$, $g(x)$, or simply gx as a shorthand for $\rho_g(x)$. Note that $gx \in X$. With this notation, the fact that ρ is a homomorphism gets hidden in an "associative law": if $g, h \in G$ and $x \in X$, then

$$(gh)x = g(hx).$$

Compare this with the literal, unpacked expression

$$\rho_{gh}(x) = (\rho_g \circ \rho_h)(x) = \rho_g(\rho_h(x)).$$

Thus we may simply write ghx without any parentheses.

DEFINITION 2.2
If X is a G-set, and $d = |X|$, then we say d is the **degree** of the representation.

DEFINITION 2.3
We say X is a **faithful** G-set if the homomorphism $\rho : G \to S_X$ is injective. In other words, X is faithful when $gx = x$ for all $x \in X$ implies $g = 1$.

Let's consider some examples.

EXAMPLE 2.4
Let $C_d = \langle \sigma \rangle$ be the cyclic group of order d and let $(12 \cdots d) \in S_d$ be a d-cycle. Then there is a unique homomorphism $\rho : C_d \to S_d$ such that $\rho(\sigma) = (12 \cdots d)$. This represents C_d as a particular d-cycle of the set $\{1, 2, \ldots, d\}$. This is a faithful representation of degree d.

PROBLEM 2.1

1. Can you find a degree 3 permutation representation of C_6?

   SOLUTION
   There are a few possible ways to do this. The most straightforward example is the unique homomorphism $\rho : C_6 \to S_3$ which sends σ to (123). Checking that this is a homomorphism reduces to verifying that the defining relation $\sigma^6 = 1$ holds in the image, and certainly $(123)^6 = 1$. □

2. Can you find a faithful degree 3 permutation representation of C_6? Find an example or prove it is impossible.

   SOLUTION
   This is impossible. Any faithful representation has to map σ to an element in S_3 with order 6, but S_3 does not have any elements of order 6. □

EXAMPLE 2.5

Let R be a commutative ring and let $\mathrm{Aff}(R)$ be the affine group of R. Given $\ell(x) = ax + b \in \mathrm{Aff}(R)$ and $r \in R$, the evaluation $\ell(r) = ar + b$ defines an action of $\mathrm{Aff}(R)$ on R. We call this the **natural** or **defining action** of $\mathrm{Aff}(R)$. The only detail that needs some thought is why evaluation *permutes* R. This is where it is important that the linear coefficient $a \in R^\times$ is a unit. In this example, our shorthand notation $\ell(r)$ feels quite natural. But keep in mind that the actual representation (which is hidden in the notation) turns the polynomial $\ell(x) = ax + b$ into a permutation of the (possibly infinite!) set R. Nevertheless, we do not need the full cycle description to calculate this permutation since $\ell(x)$ is given by a formula.

PROBLEM 2.2

Is the natural action of $\mathrm{Aff}(R)$ on R always faithful? Prove or find a counterexample.

SOLUTION

Yes, the natural action is always faithful. Suppose $\ell(x) = ax + b$ fixed all elements of R. Then

$$0 = \ell(0) = a0 + b = b$$

and

$$1 = \ell(1) = a1 = a.$$

Thus $\ell(x) = x$ is the identity. □

EXAMPLE 2.6

Let S_d be the dth symmetric group. By construction, S_d acts on $\{1, 2, \ldots, d\}$. The homomorphism here is the identity $1 : S_d \to S_d$.

EXAMPLE 2.7

We can often extract permutation representations of a group G naturally from more complicated objects on which G acts as symmetries. For example, let $\mathcal{T}$ denote the rotational symmetry group of a regular tetrahedron. These are rotations of $\mathbb{R}^3$ which leave a regular tetrahedron appearing unchanged. Let V, E, F be the sets of vertices, edges, and faces of the tetrahedron. Observe that every rotational symmetry of of the tetrahedron must permute the elements of the sets V, E, F. Hence V, E, F are each naturally $\mathcal{T}$-sets with degrees 4, 6, and 4 respectively.

PROBLEM 2.3

1. Is V a faithful $\mathcal{T}$-set?

SOLUTION

Yes, V is faithful. The fixed points of any nontrivial rotational symmetry belong to an axis passing through the center of mass of the regular tetrahedron. Each line through the center of mass passes through at most one vertex. Hence every nontrivial element of $\mathcal{T}$ fixes at most one element of V, hence V is faithful. □

2. Label the vertices of the regular tetrahedron by 1, 2, 3, 4 and let $\rho_V : \mathcal{T} \to S_4$ be the vertices representation of $\mathcal{T}$. Is ρ_V surjective? If so, prove it; otherwise determine the image of ρ_V.

SOLUTION

The map ρ_V is not surjective. Our solution of the previous part implies that a nontrivial element of $\mathcal{T}$ fixes at most one vertex. Hence no 2-cycle (ij) can be in the image of ρ_V. On the other hand, rotating around the vertex axes gives us each three cycle (ijk) in $\rho_V(\mathcal{T})$, and a 180°-rotation around each edge axis gives us permutations of the form $(ij)(k\ell)$. The smallest subgroup of $\mathcal{T}$ containing all these permutations is A_4. Thus $A_4 \subseteq \rho_V(\mathcal{T}) \subseteq S_4$. Since $[S_4 : A_4] = 2$ and ρ_V is not surjective, the only possibility is $\rho_V(\mathcal{T}) = A_4$. Thus $\mathcal{T} \cong A_4$. □

Does every group have a permutation representation? The answer is "Yes" for a silly reason. Note that S_1 is the trivial group with one element. Every group G has a unique homomorphism to S_1 which sends every element of G to the identity. This is called the **trivial representation**.

A more interesting question is whether every group G has a *faithful* permutation representation. Here the answer is again "Yes" but it takes more thought. The trick is to let G act on itself.

PROBLEM 2.4

Let G be a group. Let $\tau : G \to S_G$ be the function defined by $\tau_g(h) := gh$. We call τ the **left translation** map. Prove that τ is an injective homomorphism.

SOLUTION

First we show τ is a homomorphism. Let $g_1, g_2 \in G$. Then for all $h \in G$ we have

$$\tau_{g_1 g_2}(h) = (g_1 g_2)h = g_1(g_2 h) = \tau_{g_1}(g_2 h) = \tau_{g_1}(\tau_{g_2}(h)) = (\tau_{g_1} \circ \tau_{g_2})(h).$$

Hence $\tau_{g_1 g_2} = \tau_{g_1} \circ \tau_{g_2}$, which is what it means to say that τ is a homomorphism.
Next we show that τ is faithful Suppose $g \in \ker(\tau)$. Then $\tau_g(h) = h$ for all $h \in G$, and in

SOLUTION (CONTINUED)

particular this holds for $h = 1$, so

$$1 = \tau_g(1) = g1 = g.$$

Thus $\ker(\tau)$ is trivial, which implies that τ is injective. □

The result you proved in Problem 2.4 is known as Cayley's Theorem. This shows that *every* group may be embedded into a symmetric group. Hence no matter how abstract a group G is, no matter how weird the operation, Cayley's theorem implies we can always make G concrete by realizing it as a subgroup of some symmetric group. Hence the weird operation becomes composition of functions. This is a philosophically important result as it shows us that all abstract groups are, in a sense "real".

It is also practical: Say you wanted to implement groups in a computer. There is no way you will be able to dream up all the possible kinds of operations that group may have in advance. But Cayley's theorem tells us that you don't need to. We can just implement permutations and their composition, and then every group may be constructed in that framework.

However, on the flip-side, the left translation embedding of a group G into S_G can be incredibly inefficient. Note that $|S_G| = |G|!$ which is *way* bigger than G itself. So for large groups, this representation is impractical. Thus, for the sake of efficiency, it is often desirable to find a low-degree faithful permutation representation of G. For example, left-translation embeds S_d into S_{S_d}, which has order $d!!$. However, by construction S_d embeds into itself via the identity, which is a faithful degree d representation.

2.2 · Maps Between G-Sets

We have now defined our primary objects of interest: G-sets. When starting to build a new theory, the next step is to pin down the right kinds of maps between objects. The goal is to find maps which respect the essential structure of your objects. The general name for these kinds of structural maps is **morphism**. Along these lines, we also want to establish what it means for two G-sets to be *essentially the same* or **isomorphic**. If two G-sets X and Y are isomorphic, they should be in some sense fundamentally the same even if they are superficially different.

PROBLEM 2.5

Let G be a group and let X, Y be G-sets.

1. How should we define morphisms $f : X \to Y$? Explain why your notion is the right one.

SOLUTION

DEFINITION 2.8

Let G be a group and let X, Y be G-sets. Then a **G-map** or **morphism** between X and Y is a function $f : X \to Y$ such that for all $g \in G$ and $x \in X$ we have

$$f(gx) = gf(x). \tag{2.2.1}$$

The essential feature of a G-set is how the group G acts on it via permutations. So a map between G-sets needs to relate the G-action on one set with the G-action on the other. The identity (2.2.1) exactly captures functions between X and Y which convert the G-action of X into that of Y. □

2. Suppose X, Y, Z are G-sets and let $f_1 : Y \to Z$ and $f_2 : X \to Y$ be morphisms (following your definition). Prove that $f_1 f_2 : X \to Z$ is a morphism and prove that the identity function $1_X : X \to X$ defined by $1_X(x) = x$ is a morphism. (If you cannot prove this, then you should rethink your definition.)

SOLUTION

Say $g \in G$ and $x \in X$. Then using our definition of G-map we have

$$f_1 f_2(gx) = f_1(gf_2(x)) = gf_1 f_2(x).$$

Hence $f_1 f_2$ is also a G-map. Finally note that using the definition of the identity map we have

$$1_X(gx) = gx = g1_X(x).$$

Thus 1_X is a G-map. □

3. How should we define what it means for two G-sets to be isomorphic? Explain why your notion is the right one.

SOLUTION

DEFINITION 2.9

Let G be a group and let X, Y be G-sets. Then we say X and Y are **isomorphic** if there exists a bijective G-map $f : X \to Y$ and in this case call f an **isomorphism**.

Any isomorphism between G-sets X and Y should also be an isomorphism of sets, hence must be a bijection. The essence of a G-set is its G-action, hence our isomorphisms need

SOLUTION (CONTINUED)

to be G-maps. Our claim is that these two features are all we need. □

4. Give an example of two G-sets which are isomorphic while not literally equal.

SOLUTION

For example, let $C_3 = \langle \sigma \rangle$ be a cyclic group of order 3. We define two actions of C_3 on $X := \{1, 2, 3\}$ where $\rho_1 : C_3 \to S_3$ is given by $\rho_1(\sigma) = (123)$ and $\rho_2 : C_3 \to S_3$ is given by $\rho_2(\sigma) = (132)$. These are not literally the same C_3-sets since the generator σ acts via different permutations. But consider the permutation $f : X \to X$ defined in cycle notation by $f = (23)$. Then

$$f\sigma = (23)(123) = (13) = (132)(23) = \sigma f.$$

Hence $f\sigma^k = \sigma^k f$ for all $k \in \mathbb{Z}$ and thus for any $x \in X$ we have $f(\sigma^k(x)) = \sigma^k(f(x))$. Thus f is an isomorphism between these two C_3-actions on X. Both C_3-actions involve C_3 acting by a 3-cycle on a set with three elements. These actions only differ in the labels we put on the elements of the set, and our notion of isomorphism captures that. □

5. Find an example of a group G and two nontrivial G-sets X, Y for which there exists a morphism $f : X \to Y$ which is not an isomorphism.

SOLUTION

There are many possible examples; we will recycle some examples from earlier in this problem set. Let $G = C_6 = \langle \sigma \rangle$, let $X = \{1, 2, \ldots, 6\}$ and let $Y = \{1, 2, 3\}$. Let σ act on X by $(12 \cdots 6)$ and let σ act on Y by (123). Consider the map $f : X \to Y$ defined by $f(x) := x \bmod 3$ which returns the element of Y congruent to x modulo 3. So, for example, $f(1) = f(4) = 1$. Note that σ acts by $\sigma(x) = x + 1 \bmod 6$ on X and $\sigma(x) = x + 1 \bmod 3$ on Y. Since reduction modulo 3 is a homomorphism from $\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$, it follows that $f : X \to Y$ is a C_6-map. □

Note, we sometimes refer to a morphism of G-sets as a **G-equivariant map**.