MATH 301 · FALL 2026

# Problem Set 3: Orbits and Stabilizers

**DUE** Friday, September 18th by 11:59pm

*Of course it is exhausting, having to reason all the time*
*in a universe which wasn't meant to be reasonable.*

From *Breakfast of Champions* by Kurt Vonnegut

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

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## 3.1 · Orbits and Coproducts

A standard method in mathematics and science for trying to understand a curious object is to break it down into its fundamental components. For example, we factor numbers into primes and decompose molecules into atoms (which we then smash apart in massive colliders). The same principle applies to $G$-sets. The following terms help us put a name to the fundamental components of $G$-sets.

**DEFINITION 3.1**
Let $G$ be a group, let $X$ be a $G$-set, and let $x \in X$. The **orbit** of $x$ is the set

$$Gx := \{gx : g \in G\} \subseteq X,$$

of all elements of $X$ which may be reached from $x$ by applying elements of $G$. We say $X$ is **transitive** if $X = Gx$ for some $x \in X$.

**PROBLEM 3.1**

Let $G$ be a group and let $X$ be a $G$-set. Define the relation $\sim$ on $X$ by $x \sim y$ if there is an element $g \in G$ such that $y = gx$.

1. (Reflexive) Prove that $x \sim x$ for all $x \in X$.
2. (Symmetric) Prove that $x \sim y$ implies $y \sim x$ for all $x, y \in X$.
3. (Transitive) Prove that $x \sim y$ and $y \sim z$ together imply $x \sim z$ for all $x, y, z \in X$.
4. (Equivariance) Prove that $x \sim y$ implies $gx \sim gy$ for all $g \in G$ and $x, y \in X$.
5. Prove that the following are equivalent

(a) $x \sim y$,

(b) $y \in Gx$,

(c) $Gx = Gy$.

A reflexive, symmetric, transitive relation is called an **equivalence relation**. Equivalence relations behave essentially like equality; they measure some sense of "sameness". A collection of elements which are all equivalent to one another under an equivalence is called an **equivalence class**. In Problem 3.1 you showed that $G$-orbits are equivalence classes in $X$. Thus any two elements of an orbit generate the entire orbit. By construction, an orbit is closed under the action of $G$. Hence orbits are themselves $G$-sets which cannot be be broken into smaller pieces.

Problem 3.1 implies that orbits are either identical or they are disjoint. Hence every $G$-set may be expressed as a union of orbits. We may also formally construct new $G$-sets by combining various orbits. First, a digression on *disjoint unions*.

If $X$ and $Y$ are sets, then their union $X \cup Y$ is the smallest set that contains both $X$ and $Y$. When $X$ and $Y$ have some overlap, we may have $|X \cup Y| < |X| + |Y|$. There is a closely related operation on sets called **disjoint union** and denoted $X \sqcup Y$ which fixes this issue. The set $X \sqcup Y$ contains all the elements of $X$ and $Y$ together with tags that remember which set each element came from. These "tags" mean that even if the sets had some element in common, that will element will appear twice in the disjoint union, once with each label. We may construct $X \sqcup Y$ in several ways, but I like to think of it as the set of all elements of the form $a_X$ or $b_Y$ where $a \in X$ and $b \in Y$ (the subscripts are the tags). One nice feature of the disjoint union is that $|X \sqcup Y| = |X| + |Y|$ is always true.[1]

**DEFINITION 3.2**
Let $G$ be a group and let $X, Y$ be $G$-sets. Then $G$ acts on $X \sqcup Y$ by the actions inherited from $X$ and $Y$. That is, every element $z \in X \sqcup Y$ belongs to either $X$ or $Y$ but not both, and $g \in G$ acts on $z$ depending on which set $z$ belongs to. We call $X \sqcup Y$ together with this action the **coproduct** or **sum** of $X$ and $Y$. We also write $X + Y$ as an alternative notation for $X \sqcup Y$.

Problem 3.1 implies the following result.

**PROPOSITION 3.3**
Let $G$ be a group and let $X$ be a $G$-set. Then $X$ is isomorphic to a sum of transitive $G$-sets in an essentially unique way.

## 3.2 · Stabilizers and Coset Actions

Proposition 3.3 reduces the problem of classifying $G$-sets to the classification of transitive $G$-sets. The key to understanding the transitive $G$-sets lies in the concept of a *stabilizer*.

[1] Even when $X$ and $Y$ are infinite and we interpret the equation appropriately.

**DEFINITION 3.4**

Let $G$ be a group, let $X$ be a $G$-set, and let $x \in X$. The **stabilizer** of $x$ is the set

$$G_x := \{g \in G : gx = x\} \subseteq G,$$

of all elements of $G$ which leave $x$ fixed.

**PROBLEM 3.2**

Let $G$ be a group, let $X$ be a $G$-set, and let $x \in X$. Prove that the stabilizer $G_x$ is a subgroup of $G$.

Different points in an orbit can have different stabilizers, but they are all related to one another.

**PROBLEM 3.3**

Let $G$ be a group and let $X$ be a transitive $G$-set. If $x, y \in X$, what is the relationship between $G_x$ and $G_y$? In this circumstance, $y = gx$ for some $g \in G$. Try to phrase the relationship in terms of $g$.

Thus every choice of element in a $G$-set $X$ gives rise to a corresponding subgroup $G_x$. Flipping this around we may ask: Which subgroups $H \subseteq G$ are stabilizers of an element in some $G$-set? It turns out that every subgroup is the stabilizer for some action.

**PROBLEM 3.4**

Let $G$ be a group and let $H \subseteq G$ be a subgroup. Define a transitive action of $G$ on the set of left cosets $G/H$ such that $G_H = H$.

To what extent does the stabilizer of an element in a transitive $G$-set determine that $G$-set?

**PROBLEM 3.5**

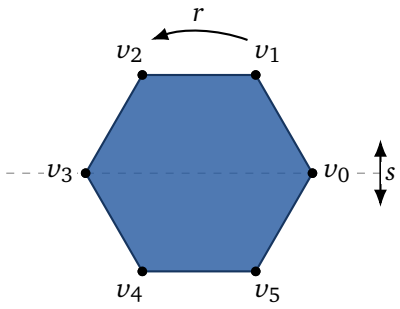
Let $G$ be a group and let $H \subseteq G$ be a subgroup. Classify up to $G$-set isomorphism all of the transitive $G$-sets $X$ which have an element $x \in X$ with $G_x = H$. If $G$ is finite, what does $G_x = H$ tell us about the size of $X$?

## 3.3 · Examples

Let's put our theory of orbits and stabilizers into action.

**PROBLEM 3.6**

The dihedral group $D_6$ naturally acts on a regular hexagon and preserves its special features. This gives us induced actions on the vertices and edges of the hexagon.




1. What is the stabilizer of $v_i$ for $0 \leq i \leq 5$?
2. What is the stabilizer of the edge $\overline{v_0 v_1}$?

## PROBLEM 3.7

Let $p$ be a prime and let $\mathbb{F}_p$ be the field with $p$ elements. Recall that $\mathrm{GL}_2(\mathbb{F}_p)$ is the group of $2 \times 2$ invertible matrices with entries in $\mathbb{F}_p$. The group $\mathrm{GL}_2(\mathbb{F}_p)$ naturally acts on the $\mathbb{F}_p$-vector space $\mathbb{F}_p^2$. Let's view $\mathbb{F}_p^2$ as a space of 2-dimensional column vectors.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

1. Prove that $\mathrm{GL}_2(\mathbb{F}_p)$ acts transitively on the nonzero vectors in $\mathbb{F}_p^2$.
2. Determine the stabilizer of $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is the order of the stabilizer?
3. What is the order of $\mathrm{GL}_2(\mathbb{F}_p)$?