MATH 301 · FALL 2026

Problem Set 3: Orbits and Stabilizers

SOLUTIONS

Of course it is exhausting, having to reason all the time
in a universe which wasn't meant to be reasonable.

From *Breakfast of Champions* by Kurt Vonnegut

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

3.1 · Orbits and Coproducts

A standard method in mathematics and science for trying to understand a curious object is to break it down into its fundamental components. For example, we factor numbers into primes and decompose molecules into atoms (which we then smash apart in massive colliders). The same principle applies to G-sets. The following terms help us put a name to the fundamental components of G-sets.

DEFINITION 3.1
Let G be a group, let X be a G-set, and let $x \in X$. The **orbit** of x is the set

$$Gx := \{gx : g \in G\} \subseteq X,$$

of all elements of X which may be reached from x by applying elements of G. We say X is **transitive** if $X = Gx$ for some $x \in X$.

PROBLEM 3.1

Let G be a group and let X be a G-set. Define the relation $\sim$ on X by $x \sim y$ if there is an element $g \in G$ such that $y = gx$.

1. (Reflexive) Prove that $x \sim x$ for all $x \in X$.

   SOLUTION
   Since $1x = x$ for all $x \in X$ it follows that $x \sim x$. □

2. (Symmetric) Prove that $x \sim y$ implies $y \sim x$ for all $x, y \in X$.

SOLUTION

If $x \sim y$, then there is some $g \in G$ such that $y = gx$. Thus $x = g^{-1}y$ and $g^{-1} \in G$ implies that $y \sim x$. □

3. (Transitive) Prove that $x \sim y$ and $y \sim z$ together imply $x \sim z$ for all $x, y, z \in X$.

SOLUTION

Suppose that $y = hx$ and $z = gy$. Then substitution yields

$$z = gy = g(hx) = (gh)x,$$

and thus $gh \in G$ implies that $x \sim z$. □

4. (Equivariance) Prove that $x \sim y$ implies $gx \sim gy$ for all $g \in G$ and $x, y \in X$.

SOLUTION

Let $h \in G$ be such that $y = hx$. Then

$$gy = ghx = (ghg^{-1})gx.$$

Thus $gx \sim gy$. □

5. Prove that the following are equivalent

   (a) $x \sim y$,

   (b) $y \in Gx$,

   (c) $Gx = Gy$.

SOLUTION

If $x \sim y$, then $y = gx$ by definition. Hence $y \in Gx$. Conversely, if $y \in Gx$, then $y = gx$ for some $g \in G$ which implies $x \sim y$. Thus (a) and (b) are equivalent.
If $x \sim y$, then $y = hx$ for some $h \in G$. If $gy \in Gy$, then $gy = ghx \in Gx$. Thus $Gy \subseteq Gx$. Since $\sim$ is symmetric, we also have $y \sim x$ and thus the same argument implies $Gx \subseteq Gy$. Therefore $x \sim y$ implies $Gx = Gy$. Conversely, if $Gx = Gy$, then $1 \in G$ implies that

$$y = 1y \in Gy = Gx.$$

Hence $y \in Gx$. □

A reflexive, symmetric, transitive relation is called an **equivalence relation**. Equivalence relations

behave essentially like equality; they measure some sense of "sameness". A collection of elements which are all equivalent to one another under an equivalence is called an **equivalence class**. In Problem 3.1 you showed that G-orbits are equivalence classes in X. Thus any two elements of an orbit generate the entire orbit. By construction, an orbit is closed under the action of G. Hence orbits are themselves G-sets which cannot be be broken into smaller pieces.

Problem 3.1 implies that orbits are either identical or they are disjoint. Hence every G-set may be expressed as a union of orbits. We may also formally construct new G-sets by combining various orbits. First, a digression on *disjoint unions*.

If X and Y are sets, then their union $X \cup Y$ is the smallest set that contains both X and Y. When X and Y have some overlap, we may have $|X \cup Y| < |X| + |Y|$. There is a closely related operation on sets called **disjoint union** and denoted $X \sqcup Y$ which fixes this issue. The set $X \sqcup Y$ contains all the elements of X and Y together with tags that remember which set each element came from. These "tags" mean that even if the sets had some element in common, that will element will appear twice in the disjoint union, once with each label. We may construct $X \sqcup Y$ in several ways, but I like to think of it as the set of all elements of the form a_X or b_Y where $a \in X$ and $b \in Y$ (the subscripts are the tags). One nice feature of the disjoint union is that $|X \sqcup Y| = |X| + |Y|$ is always true.[1]

DEFINITION 3.2
Let G be a group and let X, Y be G-sets. Then G acts on $X \sqcup Y$ by the actions inherited from X and Y. That is, every element $z \in X \sqcup Y$ belongs to either X or Y but not both, and $g \in G$ acts on z depending on which set z belongs to. We call $X \sqcup Y$ together with this action the **coproduct** or **sum** of X and Y. We also write $X + Y$ as an alternative notation for $X \sqcup Y$.

Problem 3.1 implies the following result.

PROPOSITION 3.3
Let G be a group and let X be a G-set. Then X is isomorphic to a sum of transitive G-sets in an essentially unique way.

3.2 · Stabilizers and Coset Actions

Proposition 3.3 reduces the problem of classifying G-sets to the classification of transitive G-sets. The key to understanding the transitive G-sets lies in the concept of a *stabilizer*.

DEFINITION 3.4
Let G be a group, let X be a G-set, and let $x \in X$. The **stabilizer** of x is the set

$$G_x := \{g \in G : gx = x\} \subseteq G,$$

of all elements of G which leave x fixed.

[1] Even when X and Y are infinite and we interpret the equation appropriately.

PROBLEM 3.2

Let G be a group, let X be a G-set, and let $x \in X$. Prove that the stabilizer G_x is a subgroup of G.

SOLUTION

We show G_x is closed, contains inverses, and contains the identity.

1. If $g, h \in G_x$, then using the definition of the stabilizer we have

$$(gh)x = g(hx) = gx = x.$$

   Thus $gh \in G_x$.

2. If $g \in G_x$, then $x = gx$. Multiplying both sides by g^{-1} we get

$$g^{-1}x = g^{-1}gx = 1x = x.$$

   Thus $g^{-1} \in G_x$.

3. By definition, $1x = x$, hence $1 \in G_x$.

Therefore G_x is a subgroup. □

Different points in an orbit can have different stabilizers, but they are all related to one another.

PROBLEM 3.3

Let G be a group and let X be a transitive G-set. If $x, y \in X$, what is the relationship between G_x and G_y? In this circumstance, $y = gx$ for some $g \in G$. Try to phrase the relationship in terms of g.

SOLUTION

If $y = gx$, then the stabilizers of x and y are conjugates, $G_y = gG_xg^{-1}$. Let's prove it: Suppose $h \in G_x$. Then $x = g^{-1}y$ and

$$(ghg^{-1})y = g(h(g^{-1}y)) = g(hx) = gx = y.$$

Thus $y = gx$ implies $gG_xg^{-1} \subseteq G_y$.
We also have $x = g^{-1}y$ which by the same reasoning gives us $g^{-1}G_yg \subseteq G_x$. Conjugating this containment by g we have $G_y \subseteq gG_xg^{-1}$, which completes the proof that $G_y = gG_xg^{-1}$. □

Thus every choice of element in a G-set X gives rise to a corresponding subgroup G_x. Flipping this around we may ask: Which subgroups $H \subseteq G$ are stabilizers of an element in some G-set? It turns out that every subgroup is the stabilizer for some action.

PROBLEM 3.4

Let G be a group and let $H \subseteq G$ be a subgroup. Define a transitive action of G on the set of left cosets G/H such that $G_H = H$.

SOLUTION

There are several possible actions, but only one that will give you the right stabilizer and that is simply left multiplication. Namely, if $kH \in G/H$ and $g \in G$, then

$$g(kH) := (gk)H.$$

This action is transitive by the definition of G/H. The stabilizer of H is, by definition

$$G_H = \{g \in G : gH = H\} = H. \quad \square$$

To what extent does the stabilizer of an element in a transitive G-set determine that G-set?

PROBLEM 3.5

Let G be a group and let $H \subseteq G$ be a subgroup. Classify up to G-set isomorphism all of the transitive G-sets X which have an element $x \in X$ with $G_x = H$. If G is finite, what does $G_x = H$ tell us about the size of X?

SOLUTION

Suppose that X is a transitive G-set and that $x \in X$ has $G_x = H$. Consider the function $f : G/H \to X$ defined by

$$f(gH) := gx.$$

Since cosets have more than one representative, we first need to check that this function is well-defined. Suppose that $g_1H = g_2H$. Then the coset equality criteria implies that $g_1 = g_2h$ for some $h \in H$. Since $G_x = H$ we then have

$$g_1x = g_2hx = g_2x.$$

Thus f is well-defined. Next we argue that f is G-equivariant. Observe that

$$f(g_1(g_2H)) = f((g_1g_2)H) = (g_1g_2)x = g_1(g_2x) = g_1f(g_2H).$$

Hence f is a G-map.
Our assumption that X is transitive immediately implies that f is surjective. Suppose that $f(g_1H) = f(g_2H)$. Then $g_1x = g_2x$ which implies that $g_2^{-1}g_1x = x$. Hence $g_2^{-1}g_1 \in G_x = H$. Thus $g_1 = g_2h$ for some $h \in H$ and the coset equality criteria implies that $g_1H = g_2H$. Therefore f is injective. Altogether that proves f is an isomorphism of G-sets $G/H \cong X$.
An isomorphism of G-sets is, in particular, a bijection. Therefore when G is finite we have

$$|X| = |G/H| = |G|/|H|. \tag{3.2.1}$$

SOLUTION (CONTINUED)

If $x \in X$, then X transitive implies that $X = Gx$ and $H = G_x$. Then in that case (3.2.1) reduces to

$$|G| = |Gx||G_x|,$$

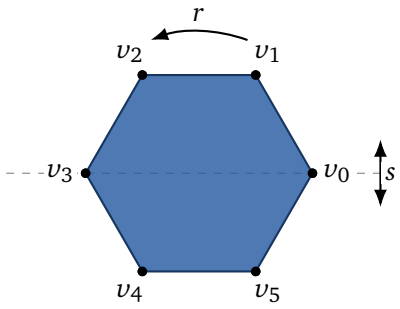
which is known as the **orbit-stabilizer theorem**. In particular, this tells us that the size of any transitive G-set is a divisor of the order of G. □

3.3 · Examples

Let's put our theory of orbits and stabilizers into action.

PROBLEM 3.6

The dihedral group D_6 naturally acts on a regular hexagon and preserves its special features. This gives us induced actions on the vertices and edges of the hexagon.



1. What is the stabilizer of v_i for $0 \leq i \leq 5$?

SOLUTION

The stabilizer of v_0 is $\langle s \rangle$. Since $r^i v_0 = v_i$, Problem 3.3 implies that the stabilizer of v_i is

$$\langle r^i s r^{-i} \rangle = \langle r^{2i} s \rangle.$$ □

2. What is the stabilizer of the edge $\overline{v_0 v_1}$?

SOLUTION

The stabilizer of $\overline{v_0 v_1}$ is $\langle rs \rangle$. To see that this is the correct reflection, note that $rsv_0 = rv_0 = r_1$ and $rsv_1 = rv_5 = v_0$, hence rs swaps v_0 and v_1. □

PROBLEM 3.7

Let p be a prime and let $\mathbb{F}_p$ be the field with p elements. Recall that $\mathrm{GL}_2(\mathbb{F}_p)$ is the group of 2×2 invertible matrices with entries in $\mathbb{F}_p$. The group $\mathrm{GL}_2(\mathbb{F}_p)$ naturally acts on the $\mathbb{F}_p$-vector space $\mathbb{F}_p^2$. Let's view $\mathbb{F}_p^2$ as a space of 2-dimensional column vectors.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

1. Prove that $\mathrm{GL}_2(\mathbb{F}_p)$ acts transitively on the nonzero vectors in $\mathbb{F}_p^2$.

SOLUTION

Suppose $v := \begin{pmatrix} a \\ c \end{pmatrix}$ is a nonzero vector. Linear algebra implies there is another nonzero vector $w := \begin{pmatrix} b \\ d \end{pmatrix}$ such that $\{v, w\}$ forms a basis of $\mathbb{F}_p^2$. Thus there is a unique invertible linear transformation A which sends $Ae_1 = v$ and $Ae_2 = w$. This linear transformation is given by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p).$$

Since $Ae_1 = v$, it follows that $\mathrm{GL}_2(\mathbb{F}_p)$ acts transitively on nonzero vectors. □

2. Determine the stabilizer of $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is the order of the stabilizer?

SOLUTION

If $A \in \mathrm{GL}_2(\mathbb{F}_p)$ stabilizes e_1, then the first column of A is e_1. The assumption that $A \in \mathrm{GL}_2(\mathbb{F}_p)$ is equivalent to the second column $\begin{pmatrix} b \\ d \end{pmatrix}$ of A being linearly independent of e_1. This reduces to $d \neq 0$ and no condition on b. Thus

$$\mathrm{GL}_2(\mathbb{F}_p)_{e_1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : d \neq 0 \right\}.$$

The order of this subgroup is $p(p-1)$. □

3. What is the order of $\mathrm{GL}_2(\mathbb{F}_p)$?

SOLUTION

The orbit-stabilizer theorem implies that

$$|\mathrm{GL}_2(\mathbb{F}_p)| = |\mathrm{GL}_2(\mathbb{F}_p)e_1||\mathrm{GL}_2(\mathbb{F}_p)_{e_1}| = (p^2 - 1)p(p-1). \quad \square$$