MATH 301 · FALL 2026

Problem Set 4: Products and Maps

DUE Friday, September 25 by 11:59pm

Nature, it seems, is the popular name
for milliards and milliards and milliards
of particles playing their infinite game
of billiards and billiards and billiards.

From *Grooks* by Piet Hein

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

4.1 · Maps Between G-sets

We now have a good structural picture of G-sets: every G-set decomposes into orbits, which are themselves transitive G-sets. If X is a transitive G-set, choosing a point $x \in X$ gives us an isomorphism $X \cong G/G_x$. Thus every G-set looks like a union of coset actions. After mastering a mathematical object, the next step is to study the maps between them.

PROBLEM 4.1

Let G be a finite group, let X, Y be G-sets, and let $\varphi : X \to Y$ be a G-map.

1. Prove that φ maps G-orbits in X to G-orbits in Y.
2. Hence it follows that we can understand all G-maps by understanding the G-maps between transitive G-sets. Let $H, K \subseteq G$ be subgroups and let $g \in G$. Determine the necessary and sufficient conditions for there to exist a G-map $\varphi : G/H \to G/K$ which sends $H \mapsto gK$.
3. When are G/H and G/K isomorphic as G-sets?
4. Recall that an **automorphism** of an object X is an isomorphism $\varphi : X \to X$. The collection $\mathrm{Aut}(X)$ of all automorphisms of X forms a group with respect to composition. Prove that the automorphism group of G/H is isomorphic to $N_G(H)/H$ where $N_G(H)$ is the **normalizer**

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

4.2 · Products of G-Sets

There are two natural operations we can use to combine G-sets to get new G-sets: sums and products. We have already discussed how every G-set is a sum of transitive G-sets. But how do products fit into this? If X, Y, Z are G-sets, then

$$X \times (Y + Z) \cong (X \times Y) + (X \times Z).$$

(do you see why?) This implies that we can reduce the analysis of products to products of transitive G-sets (do you see how this follows?)

PROBLEM 4.2

When we take the product of transitive G-sets, the result is typically not transitive anymore. The decomposition of $X \times Y$ tells us the ways in which elements of X and Y can relate to one another, up to symmetry.

1. For example, let $G = S_6$ and let $[6] = \{1, 2, \ldots, 6\}$. (In general we write $[n] := \{1, 2, \ldots, n\}$.) Consider $X = \binom{[6]}{2}$ and $Y = \binom{[6]}{3}$ where X is the set of 2 element subsets of $[6]$ and Y is the set of 3 element subsets of $[6]$. Find representatives for each orbit of S_6 on $X \times Y$. Given a pair $(x, y) \in X \times Y$, can you succinctly describe the key feature of x and y which determines which orbit it belongs to?

2. Next let $G = D_6$ and let V and E be the sets of vertices and edges of a regular hexagon. These are both D_6-sets. Draw a picture to illustrate representatives for the orbits of D_6 on $V \times E$.

PROBLEM 4.3

We know that every transitive G-set is isomorphic to G/H for some subgroup $H \subseteq G$. So if $H, K \subseteq G$, how do we decompose the G-set $G/H \times G/K$?

1. Prove that every orbit in $G/H \times G/K$ contains an element of the form (H, gK) for some $g \in G$.

2. What is the stabilizer of (H, gK)?

3. Given an element $g \in G$ and subgroups $H, K \subseteq G$, the set

$$HgK = \{hgk : h \in H, k \in K\}$$

is called a **double coset**. Let $H\backslash G/K$ denote the set of all double cosets HgK. Prove there is a bijection between $H\backslash G/K$ and the orbits of G on $G/H \times G/K$. Be sure to show that your function is well-defined!

Thus double cosets $H\backslash G/K$ precisely capture the G-orbits of $G/H \times G/K$, hence determine the decomposition of this product. Hence if X and Y are transitive G-sets, then given $(x, y) \in X \times Y$ the double cosets $G_x\backslash G/G_y$ corresponds to the fundamental ways in which elements of the orbits of x and y relate to one another up to G-symmetry.

4.3 · Burnside Semiring

The operations of addition and multiplication of G-sets behave sort of like the usual operations of addition and multiplication in a ring. One difference is that identities like

$$X \times (Y + Z) \cong (X \times Y) + (X \times Z)$$

only hold up to isomorphism, not equality. Hence we can't exactly make a ring out of G-sets. Another issue is that negatives don't make sense. But we can get around both these issues in the following way:

DEFINITION 4.1

Let G be a group. If X is a G-set, let $[X]$ denote the isomorphism class of X. Let $B(G)$ denote the set of isomorphism classes of G-sets. We can define addition and multiplication operations on $B(G)$ by

$$[X] + [Y] := [X + Y]$$
$$[X] \cdot [Y] := [X \times Y].$$

PROPOSITION 4.2

The set $B(G)$ forms a semiring[a] with respect to the operations of addition and multiplication defined above. We call $B(G)$ the **Burnside semiring** of G.

[a] A ring without negatives. We also have to add the axiom that $a0 = 0$ for all a since this cannot be derived without negatives.

I won't ask you to prove this since it's tedious to verify all the axioms. But let's think about this construction.

PROBLEM 4.4

1. What are the 0 and 1 elements in $B(G)$? Construct G-sets that represent them and explain why they work as additive and multiplicative identities.

2. The isomorphism classes $[G/H]$ corresponding to each conjugacy class of subgroups H form an additive basis for $B(G)$. If $G = S_3$, there are 4 conjugacy classes of subgroups represented by $H = \langle\varepsilon\rangle$, $\langle(12)\rangle$, $\langle(123)\rangle$, and S_3 itself. Write

$$1 = [S_3/S_3], \quad T_2 = [S_3/\langle(123)\rangle], \quad T_3 = [S_3/\langle(12)\rangle], \quad T_6 = [S_3/\langle\varepsilon\rangle],$$

so that T_n is the class of a transitive S_3-set with n elements, and 1 is the multiplicative identity from the previous part. Make a multiplication table which shows the product of each pair of elements in our additive basis $1, T_2, T_3, T_6$.

Let's put this table to work. If we have a set of combinatorial structures built on the labels $[3] = \{1, 2, 3\}$ (subsets, graphs, orderings, and so on), then S_3 acts on those structures by relabeling,

and two structures are "the same up to relabeling" precisely when they lie in the same orbit. Products of such S_3-sets arise whenever we consider two structures on the same set of labels at once.

PROBLEM 4.5

Let $\mathcal{P}$ be the set of all subsets of $[3]$ and let $\mathcal{G}$ be the set of all simple graphs[1] with vertex set $[3]$ (so $\mathcal{G}$ has $2^3 = 8$ elements, one for each subset of the three possible edges). Both are S_3-sets via the relabeling action. Use the notation T_2, T_3, T_6 from the previous problem.

1. Express $[\mathcal{P}]$ as a sum of basis elements of $B(S_3)$.
2. Express $[\mathcal{G}]$ as a sum of basis elements of $B(S_3)$.
3. A **graph with terminals** is a graph on $[3]$ together with a distinguished subset of its vertices (the terminals), that is, an element of $\mathcal{G} \times \mathcal{P}$. Use the multiplication table for $B(S_3)$ to express $[\mathcal{G} \times \mathcal{P}]$ as a sum of basis elements. How many graphs with terminals are there up to relabeling?

[1] A **simple graph** is a graph with no loops and at most one edge between vertices.