MATH 301 · FALL 2026

Problem Set 4: Products and Maps

SOLUTIONS

Nature, it seems, is the popular name
for milliards and milliards and milliards
of particles playing their infinite game
of billiards and billiards and billiards.

From *Grooks* by Piet Hein

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

4.1 · Maps Between G-sets

We now have a good structural picture of G-sets: every G-set decomposes into orbits, which are themselves transitive G-sets. If X is a transitive G-set, choosing a point $x \in X$ gives us an isomorphism $X \cong G/G_x$. Thus every G-set looks like a union of coset actions. After mastering a mathematical object, the next step is to study the maps between them.

PROBLEM 4.1

Let G be a finite group, let X, Y be G-sets, and let $\varphi : X \to Y$ be a G-map.

1. Prove that φ maps G-orbits in X to G-orbits in Y.

   SOLUTION

   Let $x \in X$. Since φ is a G-map, we have

   $$\varphi(Gx) = G\varphi(x).$$

   Thus φ maps orbits to orbits. □

2. Hence it follows that we can understand all G-maps by understanding the G-maps between transitive G-sets. Let $H, K \subseteq G$ be subgroups and let $g \in G$. Determine the necessary and sufficient conditions for there to exist a G-map $\varphi : G/H \to G/K$ which sends $H \mapsto gK$.

SOLUTION

If $\varphi(H) = gK$ is a G-map, then for all $a \in G$ we have

$$\varphi(aH) = a\varphi(H) = agK.$$

So there is at most one such G-map. If $h \in H$, then $hH = H$, hence applying φ we get

$$gK = \varphi(H) = \varphi(hH) = hgK.$$

This is equivalent to $g^{-1}hg \in K$ for all $h \in H$. Hence a necessary condition is that $g^{-1}Hg \subseteq K$. Running this argument backwards, we can see this condition is also sufficient. □

3. When are G/H and G/K isomorphic as G-sets?

SOLUTION

The G-sets G/H and G/K are isomorphic if and only if H and K are conjugate subgroups. If $g^{-1}Hg = K$, then we also have $H = gKg^{-1}$. Hence the previous problem shows there is a G-map $\varphi : G/H \to G/K$ determined by $\varphi(H) = gK$ and a map $\psi : G/K \to G/H$ determined by $\psi(K) = g^{-1}H$. Thus $\psi\varphi : G/H \to G/H$ is the G-map determined by

$$\psi\varphi(H) = \psi(gK) = g\psi(K) = gg^{-1}H = H.$$

The identity map $1 : G/H \to G/H$ also satisfies this, hence $\psi\varphi = 1$. The identity $\varphi\psi = 1$ follows similarly. Thus φ and ψ are inverses, implying that $G/H \cong G/K$. □

4. Recall that an **automorphism** of an object X is an isomorphism $\varphi : X \to X$. The collection $\mathrm{Aut}(X)$ of all automorphisms of X forms a group with respect to composition. Prove that the automorphism group of G/H is isomorphic to $N_G(H)/H$ where $N_G(H)$ is the **normalizer**

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

SOLUTION

A G-map $\varphi : G/H \to G/H$ is determined by where it sends H. We know that $H \mapsto g^{-1}H$ is a G-map if and only if $gHg^{-1} \subseteq H$. Since H is finite, this implies that $gHg^{-1} = H$. Thus we have a surjective homomorphism $\varphi : N_G(H) \to \mathrm{Aut}(G/H)$ given by $g \mapsto \varphi_g : G/H \to G/H$ where φ_g is determined by $\varphi_g(H) := g^{-1}H$. Note that the inverse here is important;

SOLUTION (CONTINUED)

without it we do not get a homomorphism:

$$\begin{aligned}\varphi_{g_1}\varphi_{g_2}(H) &= \varphi_{g_1}(g_2^{-1}H)\\ &= g_2^{-1}\varphi_{g_1}(H)\\ &= g_2^{-1}g_1^{-1}H\\ &= (g_1g_2)^{-1}H\\ &= \varphi_{g_1g_2}(H).\end{aligned}$$

The first isomorphism theorem then implies that $\mathrm{Aut}(G/H) \cong N_G(H)/\ker(\varphi)$. Observe that $g \in \ker(\varphi)$ if and only if φ_g is the identity automorphism of G/H, if and only if $gH = H$, if and only if $g \in H$. Hence $\ker(\varphi) = H$. Therefore

$$N_G(H)/H \cong \mathrm{Aut}(G/H). \qquad \square$$

4.2 · Products of G-Sets

There are two natural operations we can use to combine G-sets to get new G-sets: sums and products. We have already discussed how every G-set is a sum of transitive G-sets. But how do products fit into this? If X, Y, Z are G-sets, then

$$X \times (Y + Z) \cong (X \times Y) + (X \times Z).$$

(do you see why?) This implies that we can reduce the analysis of products to products of transitive G-sets (do you see how this follows?)

PROBLEM 4.2

When we take the product of transitive G-sets, the result is typically not transitive anymore. The decomposition of $X \times Y$ tells us the ways in which elements of X and Y can relate to one another, up to symmetry.

1. For example, let $G = S_6$ and let $[6] = \{1, 2, \ldots, 6\}$. (In general we write $[n] := \{1, 2, \ldots, n\}$.) Consider $X = \binom{[6]}{2}$ and $Y = \binom{[6]}{3}$ where X is the set of 2 element subsets of $[6]$ and Y is the set of 3 element subsets of $[6]$. Find representatives for each orbit of S_6 on $X \times Y$. Given a pair $(x, y) \in X \times Y$, can you succinctly describe the key feature of x and y which determines which orbit it belongs to?

SOLUTION

The orbits are determined by the size of the overlap between x and y. There are thus three orbits represented by

SOLUTION (CONTINUED)

(a) $(\{1,2\},\{3,4,5\})$

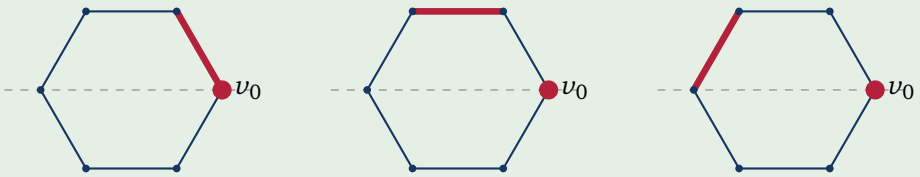
(b) $(\{1,2\},\{2,3,4\})$

(c) $(\{1,2\},\{1,2,3\})$. □

2. Next let $G = D_6$ and let V and E be the sets of vertices and edges of a regular hexagon. These are both D_6-sets. Draw a picture to illustrate representatives for the orbits of D_6 on $V \times E$.

SOLUTION

Fix a vertex v_0. Every orbit of D_6 on $V \times E$ contains a pair of the form (v_0, e), and two such pairs (v_0, e) and (v_0, e') lie in the same orbit exactly when $e' = ge$ for some g in the stabilizer of v_0. That stabilizer is $\{1, s\}$ where s is the reflection across the axis through v_0 and v_3, and s pairs up the six edges into three pairs. Hence there are three orbits, determined by how far the edge is from the vertex: the edge contains the vertex, the edge is one step away, or the edge is two steps away (adjacent to the opposite vertex).



edge contains v_0 | one step away | two steps away

Since $|V \times E| = 36$ and D_6 has order 12, each of the three orbits has exactly 12 elements, so the stabilizer of each pair (v_0, e) is trivial. □

PROBLEM 4.3

We know that every transitive G-set is isomorphic to G/H for some subgroup $H \subseteq G$. So if $H, K \subseteq G$, how do we decompose the G-set $G/H \times G/K$?

1. Prove that every orbit in $G/H \times G/K$ contains an element of the form (H, gK) for some $g \in G$.

SOLUTION

Every element of $G/H \times G/K$ looks like (g_1H, g_2K) for some $g_1, g_2 \in G$. Multiplying by g_1^{-1} gives us

$$g_1^{-1}(g_1H, g_2K) = (H, g_1^{-1}g_2K).$$

Hence let $g := g_1^{-1}g_2$, then (H, gK) belongs to the orbit of (g_1H, g_2K). □

2. What is the stabilizer of (H, gK)?

SOLUTION

Note that g' belongs to the stabilizer of (H, gK) if and only if $(g'H, g'gK) = (H, gK)$, which is equivalent to $g' \in H$ and $g^{-1}g'g \in K$. And this is equivalent to $g' \in H \cap gKg^{-1}$. Hence the stabilizer of (H, gK) is $H \cap gKg^{-1}$. □

3. Given an element $g \in G$ and subgroups $H, K \subseteq G$, the set

$$HgK = \{hgk : h \in H, k \in K\}$$

is called a **double coset**. Let $H\backslash G/K$ denote the set of all double cosets HgK. Prove there is a bijection between $H\backslash G/K$ and the orbits of G on $G/H \times G/K$. Be sure to show that your function is well-defined!

SOLUTION

Let $Y := (G/H \times G/K)/G$ denote the set of G-orbits on this product space. Let $\psi : H\backslash G/K \to Y$ be defined by $\psi(HgK) = G(H, gK)$. If $Hg_1K = Hg_2K$, then $g_2 = hg_1k$ for some $h \in H, k \in K$. Then

$$(H, g_2K) = (H, hg_1kK) = (hH, hg_1K) = h(H, g_1K).$$

Hence $G(H, g_1K) = G(H, g_2K)$, which shows that ψ is well-defined.
Part 1 shows that ψ is surjective, hence it remains to show that ψ is injective. Suppose that $G(H, g_1K) = G(H, g_2K)$. Then there exists some $g \in G$ such that

$$(H, g_2K) = g(H, g_1K) = (gH, gg_1K).$$

Comparing first coordinates we see that $g \in H$ and that $g_2K = gg_1K$. Thus the coset equality criteria implies that $g_2 = gg_1k$ for some $k \in K$. Hence

$$Hg_2K = Hgg_1kK = Hg_1K.$$

Thus ψ is a bijection. □

Thus double cosets $H\backslash G/K$ precisely capture the G-orbits of $G/H \times G/K$, hence determine the decomposition of this product. Hence if X and Y are transitive G-sets, then given $(x, y) \in X \times Y$ the double cosets $G_x\backslash G/G_y$ corresponds to the fundamental ways in which elements of the orbits of x and y relate to one another up to G-symmetry.

4.3 · Burnside Semiring

The operations of addition and multiplication of G-sets behave sort of like the usual operations of addition and multiplication in a ring. One difference is that identities like

$$X \times (Y + Z) \cong (X \times Y) + (X \times Z)$$

only hold up to isomorphism, not equality. Hence we can't exactly make a ring out of G-sets. Another issue is that negatives don't make sense. But we can get around both these issues in the following way:

DEFINITION 4.1

Let G be a group. If X is a G-set, let $[X]$ denote the isomorphism class of X. Let $B(G)$ denote the set of isomorphism classes of G-sets. We can define addition and multiplication operations on $B(G)$ by

$$[X] + [Y] := [X + Y]$$
$$[X] \cdot [Y] := [X \times Y].$$

PROPOSITION 4.2

The set $B(G)$ forms a semiring[a] with respect to the operations of addition and multiplication defined above. We call $B(G)$ the **Burnside semiring** of G.

[a] A ring without negatives. We also have to add the axiom that $a0 = 0$ for all a since this cannot be derived without negatives.

I won't ask you to prove this since it's tedious to verify all the axioms. But let's think about this construction.

PROBLEM 4.4

1. What are the 0 and 1 elements in $B(G)$? Construct G-sets that represent them and explain why they work as additive and multiplicative identities.

   SOLUTION

   The empty set $\emptyset$ represents the 0 element of $B(G)$ and the singleton set $\{*\}$ with the trivial G-action represents the 1 element. The empty set is clearly an additive identity with respect to union, and the singleton is an identity with respect to products. Finally note that $X \times \emptyset = \emptyset$ for all sets X, which shows that $[X]0 = 0$. □

2. The isomorphism classes $[G/H]$ corresponding to each conjugacy class of subgroups H form an additive basis for $B(G)$. If $G = S_3$, there are 4 conjugacy classes of subgroups represented by $H = \langle \varepsilon \rangle$, $\langle (12) \rangle$, $\langle (123) \rangle$, and S_3 itself. Write

   $$1 = [S_3/S_3], \quad T_2 = [S_3/\langle (123) \rangle], \quad T_3 = [S_3/\langle (12) \rangle], \quad T_6 = [S_3/\langle \varepsilon \rangle],$$

so that T_n is the class of a transitive S_3-set with n elements, and 1 is the multiplicative identity from the previous part. Make a multiplication table which shows the product of each pair of elements in our additive basis $1, T_2, T_3, T_6$.

SOLUTION

Concretely, 1 is a point, T_2 is S_3 acting on $\{\pm 1\}$ by sign, T_3 is the natural action on $\{1,2,3\}$, and T_6 is S_3 acting on itself by left multiplication. Our previous problem tells us how to decompose a product: the orbits of $S_3/H \times S_3/K$ correspond to double cosets $HgK \in H\backslash S_3/K$, and the orbit of (H, gK) is isomorphic to $S_3/(H \cap gKg^{-1})$. Since 1 is the multiplicative identity, we only need to compute the products of T_2, T_3, and T_6.

- *Products with T_6.* If $H = \langle\varepsilon\rangle$, then $H \cap gKg^{-1}$ is trivial for every K and g, so every orbit of $S_3 \times S_3/K$ is isomorphic to S_3 itself. If S_3/K has n elements, then counting elements gives $6n = 6 \cdot$ (number of orbits), so there are n orbits and $T_6T_n = nT_6$. Hence $T_6T_2 = 2T_6$, $T_6T_3 = 3T_6$, and $T_6^2 = 6T_6$.
- T_2^2. The subgroup $A_3 = \langle(123)\rangle$ is normal, so $A_3 \cap gA_3g^{-1} = A_3$ for every g and every orbit of $S_3/A_3 \times S_3/A_3$ is isomorphic to S_3/A_3. There are $4/2 = 2$ orbits, so $T_2^2 = 2T_2$. (Concretely, the orbits of S_3 on $\{\pm 1\}^2$ are $\{(1,1),(-1,-1)\}$ and $\{(1,-1),(-1,1)\}$.)
- T_2T_3. The intersection $\langle(123)\rangle \cap g\langle(12)\rangle g^{-1}$ is trivial by Lagrange's theorem, since the subgroups have orders 3 and 2. Thus every orbit of $\{\pm 1\} \times \{1,2,3\}$ is isomorphic to S_3, and since there are only 6 elements, there is a single orbit: $T_2T_3 = T_6$.
- T_3^2. The action of S_3 on $\{1,2,3\}^2$ has two orbits: the diagonal $\{(i,i)\}$, which is isomorphic to $\{1,2,3\}$, and the six pairs (i,j) with $i \neq j$, on which S_3 acts freely (a permutation fixing i and j also fixes the third point). Thus $T_3^2 = T_3 + T_6$. In terms of double cosets, if $H = \langle(12)\rangle$ then $H\backslash S_3/H = \{H, H(13)H\}$ with $H \cap (13)H(13)^{-1} = \langle(12)\rangle \cap \langle(23)\rangle = \langle\varepsilon\rangle$.

Putting this together, the multiplication table for $B(S_3)$ is:

$\cdot$	1	T_2	T_3	T_6
1	1	T_2	T_3	T_6
T_2	T_2	$2T_2$	T_6	$2T_6$
T_3	T_3	T_6	$T_3 + T_6$	$3T_6$
T_6	T_6	$2T_6$	$3T_6$	$6T_6$

□

Let's put this table to work. If we have a set of combinatorial structures built on the labels $[3] = \{1,2,3\}$ (subsets, graphs, orderings, and so on), then S_3 acts on those structures by relabeling, and two structures are "the same up to relabeling" precisely when they lie in the same orbit. Products of such S_3-sets arise whenever we consider two structures on the same set of labels at once.

PROBLEM 4.5

Let $\mathcal{P}$ be the set of all subsets of $[3]$ and let $\mathcal{G}$ be the set of all simple graphs[1] with vertex set $[3]$ (so $\mathcal{G}$ has $2^3 = 8$ elements, one for each subset of the three possible edges). Both are S_3-sets via the relabeling action. Use the notation T_2, T_3, T_6 from the previous problem.

1. Express $[\mathcal{P}]$ as a sum of basis elements of $B(S_3)$.

SOLUTION

The orbit of a subset $S \subseteq [3]$ is determined by $|S|$, since S_3 can move any k-element subset to any other. The subsets $\emptyset$ and $[3]$ are fixed by all of S_3, so each forms an orbit isomorphic to a point. The three 1-element subsets form one orbit, and the stabilizer of $\{1\}$ is $\langle(23)\rangle$, which is conjugate to $\langle(12)\rangle$, so this orbit is isomorphic to $S_3/\langle(12)\rangle$. The same holds for the three 2-element subsets (take complements). Therefore

$$[\mathcal{P}] = 1 + T_3 + T_3 + 1 = 2 + 2T_3. \qquad \square$$

2. Express $[\mathcal{G}]$ as a sum of basis elements of $B(S_3)$.

SOLUTION

Up to relabeling there are four graphs on $[3]$, determined by the number of edges: the empty graph, a single edge, a path with two edges, and the triangle. The empty graph and the triangle are fixed by S_3, giving two orbits isomorphic to a point. The graph with the single edge $\{1, 2\}$ has stabilizer $\langle(12)\rangle$, so the three single-edge graphs form an orbit isomorphic to $S_3/\langle(12)\rangle$. Likewise the path $1 - 2 - 3$ has stabilizer $\langle(13)\rangle$, so the three paths form an orbit isomorphic to $S_3/\langle(12)\rangle$. Therefore

$$[\mathcal{G}] = 2 + 2T_3.$$

(Alternatively: a graph on $[3]$ is the same thing as a subset of the three edges, and S_3 permutes the edges the same way it permutes the vertices, so $\mathcal{G} \cong \mathcal{P}$ as S_3-sets.) $\square$

3. A **graph with terminals** is a graph on $[3]$ together with a distinguished subset of its vertices (the terminals), that is, an element of $\mathcal{G} \times \mathcal{P}$. Use the multiplication table for $B(S_3)$ to express $[\mathcal{G} \times \mathcal{P}]$ as a sum of basis elements. How many graphs with terminals are there up to relabeling?

SOLUTION

Using the table for $B(S_3)$ and $T_3^2 = T_3 + T_6$,

$$\begin{aligned}[\mathcal{G} \times \mathcal{P}] &= (2 + 2T_3)(2 + 2T_3) = 4 + 8T_3 + 4T_3^2 \\ &= 4 + 8T_3 + 4(T_3 + T_6) = 4 + 12T_3 + 4T_6.\end{aligned}$$

[1] A **simple graph** is a graph with no loops and at most one edge between vertices.

SOLUTION (CONTINUED)

Each basis element is a single orbit, so there are $4 + 12 + 4 = 20$ graphs with terminals up to relabeling: 4 fixed by all of S_3, 12 with orbits of size 3, and 4 with orbits of size 6. As a check on the sizes, $4 \cdot 1 + 12 \cdot 3 + 4 \cdot 6 = 64 = 8 \cdot 8 = |\mathcal{G} \times \mathcal{P}|$.
As a check, we can list the orbits directly. First choose a graph Γ from each of the four orbits of $\mathcal{G}$, and then the orbits of $\mathcal{G} \times \mathcal{P}$ containing (Γ, S) correspond to the orbits of the stabilizer $\mathrm{Aut}(\Gamma)$ on subsets $S \subseteq [3]$. The size of the orbit of (Γ, S) in $\mathcal{G} \times \mathcal{P}$ is $6/|\mathrm{Aut}(\Gamma) \cap \mathrm{Stab}(S)|$.

- *Empty graph*, $\mathrm{Aut}(\Gamma) = S_3$: the orbits of subsets are determined by size, giving 4 orbits. For $S = \emptyset, [3]$ the pair (Γ, S) is fixed by S_3 (orbit size 1); for $|S| = 1, 2$ the orbit has size 3.
- *Triangle*, $\mathrm{Aut}(\Gamma) = S_3$: exactly the same, 2 orbits of size 1 and 2 of size 3.
- *Single edge* $\{1, 2\}$, $\mathrm{Aut}(\Gamma) = \langle (12) \rangle$: the subsets $\emptyset$, $\{3\}$, $\{1, 2\}$, $\{1, 2, 3\}$ are each fixed by (12) and give 4 orbits of size 3, while $\{1\} \sim \{2\}$ and $\{1, 3\} \sim \{2, 3\}$ give 2 orbits of size 6.
- *Path* $1 - 2 - 3$, $\mathrm{Aut}(\Gamma) = \langle (13) \rangle$: the same count, 4 orbits of size 3 and 2 orbits of size 6.

Altogether we find 4 orbits of size 1, $4 + 4 + 4 = 12$ orbits of size 3, and $2 + 2 = 4$ orbits of size 6, for a total of 20, exactly as predicted by $4 + 12T_3 + 4T_6$. □