MATH 301 · FALL 2026

Problem Set 6: Sum Decompositions

DUE Friday, October 30th by 11:59pm

> *First, there's the room you can see through the glass—that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair—all but the bit behind the fireplace. Oh! I do so wish I could see* that *bit!*
>
> From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

Let's recall a definition from the end of the last problem set.

DEFINITION 6.1
Let G be a group and let V be a G-representation. A subspace $U \subseteq V$ is called a G**-invariant subspace** or a G**-subrepresentation** if $gU \subseteq U$ for all $g \in G$.

In Problem 5.5 we analyzed the S_3-invariant subspaces of $V := \mathbb{C}[3] \cong \mathbb{C}^3$ where $[3] = \{1, 2, 3\}$ is an S_3-set. There are precisely 4 of them:

1. $\mathbf{0} := \{0\}$,
2. $V_1 := \text{Span}((1, 1, 1))$,
3. $V_2 := \text{Span}((1, -1, 0), (1, 0, -1))$,
4. V,

where we use $\text{Span}(v_1, \ldots, v_k)$ to denote the span of the v_i. The $\mathbf{0}$ and V cases are boring, every representation V has $\mathbf{0}$ and V as invariant subspaces. The V_1 and V_2 are more interesting. For one, notice that they are orthogonal to one another.

The fact that these spaces are orthogonal to one another implies that the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, -1, 0)$, and $v_3 = (1, 0, -1)$ form a basis $\mathfrak{B}$ for V. This is a special basis for our group action. If we express the elements of S_3 in terms of the basis $\mathfrak{B}$ they have a special form. For example, consider the action of (123). To find the matrix representation $[(123)]_{\mathfrak{B}}$ with respect to the basis $\mathfrak{B}$ note that

$$\begin{aligned}(123)v_1 &= (123)(1, 1, 1) = (1, 1, 1) = v_1\\ (123)v_2 &= (123)(1, -1, 0) = (0, 1, -1) = v_3 - v_2\\ (123)v_3 &= (123)(1, 0, -1) = (-1, 1, 0) = -v_2.\end{aligned}$$

Hence

$$[(123)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

I'll let you work out the rest.

PROBLEM 6.1

Work out the matrix representation $[g]_{\mathfrak{B}}$ for each $g \in S_3$. What do you notice about the form of these matrices?

The special form the matrix representations of S_3 take with respect to the basis $\mathfrak{B}$ is a reflection of the structure of V as an S_3-representation. To understand this we need to review the notion of a *direct sum* of vector spaces.

6.1 · Direct Sums

If U and V are vector spaces, then their **direct sum** $U \oplus V$ is simply the Cartesian product $U \times V$ with the vector space operations defined componentwise.[1] We typically write $u \oplus v$ to denote (u, v) in order to remind ourselves to think of $u \oplus v$ as a formal sum of two vectors. Note that the vector spaces U and V could be totally unrelated so that there is no natural way to add vectors between U and V. The direct sum allows us to formally combine these into one vector space where we use the special symbol $\oplus$ to denote the formal sum combining these disparate vectors.

If U and V are G-representations, say with actions $\rho : G \to \mathrm{Aut}(U)$ and $\rho' : G \to \mathrm{Aut}(V)$, then $U \oplus V$ also forms a representation of G with action $\rho \oplus \rho' : G \to \mathrm{Aut}(U \oplus V)$ defined by

$$(\rho \oplus \rho')_g(u \oplus v) := \rho_g(u) \oplus \rho'_g(v).$$

Or with the actions left implicit, this simplifies to

$$g(u \oplus v) = gu \oplus gv.$$

Hence G acts separately on U and V with no interaction between them.

Before going further, let's say what it means for two G-representations to be isomorphic. The definition follows the same pattern as G-maps of G-sets and R-linear maps of R-modules: the map should commute with the action.

[1] Important caveat: There is an important difference between Cartesian products and direct sums of *infinitely many* vector spaces, but for *finitely many* vector spaces, the two concepts are the same.

DEFINITION 6.2

Let U and V be G-representations. A linear map $\varphi : U \to V$ is called **G-linear** (or a **G-map**) if

$$\varphi(gu) = g\varphi(u)$$

for all $g \in G$ and $u \in U$. A **G-linear isomorphism** is a bijective G-linear map. If a G-linear isomorphism $U \to V$ exists we say U and V are **isomorphic as G-representations** and write $U \cong V$.

Just as for R-modules, the inverse of a bijective G-linear map is automatically G-linear, so a G-linear isomorphism has a G-linear inverse.

PROBLEM 6.2

1. Let U, V be vector spaces. Suppose that $\mathfrak{B}_U := \{u_1, \ldots, u_m\}$ is a basis for U and $\mathfrak{B}_V := \{v_1, \ldots, v_n\}$ is a basis for V. Prove that

   $$\mathfrak{B}_{U\oplus V} := \{u_1 \oplus 0, \ldots, u_m \oplus 0, 0 \oplus v_1, \ldots, 0 \oplus v_n\}$$

   is a basis for $U \oplus V$.

2. Now suppose that U and V are both G-representations. Given $g \in G$, let $[g]_U$ and $[g]_V$ denote the matrix representations for g acting on U and V with respect to the bases $\mathfrak{B}_U$ and $\mathfrak{B}_V$, respectively. Similarly, let $[g]_{U\oplus V}$ denote the matrix representation of g with respect to $\mathfrak{B}_{U\oplus V}$. Prove that

   $$[g]_{U\oplus V} = \begin{bmatrix} [g]_U & 0 \\ 0 & [g]_V \end{bmatrix},$$

   where the 0 here represent rectangular matrices of zeros. Matrices in this form are called **block diagonal** matrices.

While direct sums may be used to combine G-representations, we more often use them to decompose a given G-representation. Let W be a vector space and suppose that $U, V \subseteq W$ are subspaces. We define the **internal sum** $U + V \subseteq W$ to be the subspace

$$U + V = \{u + v : u \in U, v \in V\}.$$

Note that since U and V live inside a common vector space W, it does make sense to add vectors from U and V.

PROBLEM 6.3

Suppose that $U, V \subseteq W$ are subspaces such that $U + V = W$ and $U \cap V = \mathbf{0}$. Prove that $W \cong U \oplus V$. These conditions are called the **internal direct sum criteria**.

If W is a G-representation and $U, V \subseteq W$ are subrepresentations satisfying the internal direct sum criteria, then we also have $W \cong U \oplus V$ as G-representations. When studying a representation of G, we often try to decompose it as a sum of smaller, simpler representations. This is the linear representation analog of studying a group action by decomposing the action into orbits.

PROBLEM 6.4

Let X and Y be G-sets. Recall that $X+Y$ is the disjoint union of X and Y. Prove that $\mathbb{C}(X+Y) \cong \mathbb{C}X \oplus \mathbb{C}Y$. **Hint:** Without loss of generality you may assume that $X, Y \subseteq X+Y$. This will allow you to use the internal direct sum criteria.

As we recursively decompose a representation, eventually we reach invariant subspaces which cannot decompose any further. These are called *irreducible representations* or *irreps* for short.

DEFINITION 6.3

A nonzero G-representation W is said to be **irreducible** if $W \ncong U \oplus V$ for any U, V with positive dimension.

For example, the plane $V_2 = \text{Span}((1,-1,0),(1,0,-1))$ is an irreducible S_3-representation: If not, then $V_2 \cong U_1 \oplus U_2$ where the U_i both have dimension 1. That would mean V_2 contains two S_3-invariant lines. However note that (123) acts by a 120° rotation of the plane, hence does not leave any lines invariant. Thus no U_1 and U_2 exist.

Just as every G-set breaks down into a disjoint union of transitive G-sets, every representation breaks down into a direct sum of irreducibles.

PROBLEM 6.5

Prove that every nonzero finite dimensional representation V is isomorphic to a finite direct sum of irreducible representations.

We will ultimately show that every finite dimensional representation has an essentially unique decomposition into a direct sum of irreps. But first some preliminary work. For instance, how do we know whether a given representation W is irreducible? If $W \cong U \oplus V$, that implies W has proper invariant subspaces isomorphic to U and V, so I could start by looking for an invariant subspace U. Say we find an invariant subspace U: will there always exist a complementary invariant subspace V such that $W \cong U \oplus V$? We call an invariant subspace $V \subseteq W$ such that $U + V = W$ and $U \cap V = \mathbf{0}$ a **complementary subspace** of U.

PROBLEM 6.6

Let $C_\infty = \langle g \rangle$ be the infinite cyclic group. An n-dimensional representation of C_∞ is equivalent to the choice of an invertible $n \times n$ matrix. For example, consider the 2-dimensional representation $W = \mathbb{C}^2$ of C_∞ given by

$$g \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Prove that $U = \text{Span}(e_1)$ is a C_∞-invariant subspace and that U has no complementary invariant subspace.

This problem shows that complementary invariant subspaces do not always exist. However, the issue in this example stems entirely from the fact that C_∞ is an infinite group. The situation with finite groups works out quite neatly. To explain, we need to first discuss inner products.

6.2 · Inner Products

Given two vectors $v = (a_1, \ldots, a_n), w = (b_1, \ldots, b_n) \in \mathbb{R}^n$, their dot product is

$$v \cdot w := \sum_{i=1}^{n} a_i b_i.$$

The dot product is used to measure the correlation between two vectors. This can be seen from the formula

$$v \cdot w = |v||w| \cos\theta,$$

where $|v|$ is the magnitude of v and θ is the angle between v and w. The idea is that if the angle between v and w is small, then $\cos(\theta)$ is close to one and the dot product is essentially just the product of their magnitudes. But if v and w are close to being orthogonal, then $\cos(\theta)$ is nearly zero.

The familiar dot product generalizes to the notion of an *inner product*.

DEFINITION 6.4

Let V be a $\mathbb{C}$-vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called an **inner product** if

1. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$,
2. $\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$ for all $v_1, v_2, w \in V$ and all $c_1, c_2 \in \mathbb{C}$,
3. $\langle v, v \rangle > 0$ for all nonzero $v \in V$.

A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space**.

PROBLEM 6.7

Let's review/get to know inner products. Suppose V is an inner product space.

1. Prove that $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$ (hence that the third axiom defining the inner product makes sense).
2. Prove that $\langle v, c_1 w_1 + c_2 w_2 \rangle = \overline{c_1} \langle v, w_1 \rangle + \overline{c_2} \langle v, w_2 \rangle$ for all $v, w_1, w_2 \in V$ and all $c_1, c_2 \in \mathbb{C}$.
3. Prove that $\langle 0, w \rangle = \langle v, 0 \rangle = 0$ for all $v, w \in V$.

While we cannot use a general inner product to measure angles in a complex vector space, we can still use them to detect when two vectors are orthogonal to one another: we say $v, w \in V$ are **orthogonal** if $\langle v, w \rangle = 0$.

If V is a vector space and $U \subseteq V$ is a subspace, then in general there is no canonical complementary subspace $W \subseteq V$ such that $V \cong U \oplus W$. You can always find a complement, but it will require making some choices. However, one thing an inner product gives us is canonical complements.

PROBLEM 6.8

1. Let V be a finite dimensional inner product space and let $U \subseteq V$ be a subspace. Prove that

$$U^\perp := \{v \in V : \langle u, v\rangle = 0 \text{ for all } u \in U\}$$

   is a subspace of V and that $V \cong U \oplus U^\perp$. **Hint:** Choose a basis $u_1, \ldots, u_k$ of U and consider the linear transformation $T : V \to \mathbb{C}^k$ defined by

$$T(v) := (\langle v, u_1\rangle, \ldots, \langle v, u_k\rangle).$$

   We call $U^\perp$ the **orthogonal complement** of U.

2. Now suppose that V is a G-representation and that $U \subseteq V$ is an invariant subspace. Furthermore, let's suppose that V has an inner product $\langle \cdot, \cdot \rangle$ which is G-invariant:

$$\langle gv, gw\rangle = \langle v, w\rangle$$

   for all $g \in G$ and $v, w \in V$. Prove that $U^\perp$ is also G-invariant and that $V \cong U \oplus U^\perp$ as G-representations.

3. Prove that the C_∞-representation W constructed in Problem 6.6 does not have any C_∞-invariant inner product.

Hence if V is a G-representation with a G-invariant inner product, then every subrepresentation has a complement. Thus a natural question to ask is: which representations have a G-invariant inner product? Suppose G is a finite group and that V is a G-representation with an inner product $\langle \cdot, \cdot \rangle$ (not necessarily G-invariant). We then define

$$\langle v, w\rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw\rangle.$$

PROBLEM 6.9

Prove that $\langle \cdot, \cdot \rangle_G$ is a G-invariant inner product on V.

Thus if G is finite we can take any inner product on V and average it over the group action in order to construct a G-invariant inner product. Notice that this only works if G is finite, since if G is infinite we run into convergence issues with the infinite sum and there's the issue of dividing by $|G| = \infty$. [2]

[2] We can sometimes make this work when G is infinite, but we need a topology on both G and V. When the group G is compact (like the circle group $U(1)$) we can make this work using an integral in place of the sum. Essentially everything we prove about representations of finite groups may be generalized to compact topological groups as well.

There's one last detail to check: Which G-representations have an inner product? We need one in order to average it and get a G-invariant inner product.

PROBLEM 6.10

1. Construct an explicit inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ for all $n \geq 1$.

2. Let V be an n-dimensional complex vector space and suppose that $\varphi : V \to \mathbb{C}^n$ is an isomorphism. Prove that the function
$$\langle v, w \rangle_\varphi := \langle \varphi(v), \varphi(w) \rangle$$
defines an inner product on V.

3. If G is a finite group and V is a nonzero finite dimensional G-representation, prove that V is irreducible if and only if the only G-invariant subspaces of V are $\mathbf{0}$ and V itself.

Thus, to summarize what we have now shown: If G is a finite group and V is a finite dimensional G-representation, then V has a G-invariant inner product. The irreducible representations of G are precisely those representations V whose only invariant subspaces are $\mathbf{0}$ and V itself. Every finite dimensional representation V of G may be decomposed as a sum of irreps.

This raises two natural questions. First, the proof of Problem 6.5 involved choices; is the resulting list of irreps unique? Second, how do we find the irreps of a given group, and how do we recognize them when we see them? The key to both questions is to study the G-linear maps between representations.

6.3 · Schur's Lemma

Let V and W be G-representations. We write $\mathrm{Hom}_G(V, W)$ for the set of all G-linear maps $\varphi : V \to W$. If $\varphi, \psi \in \mathrm{Hom}_G(V, W)$ and $c \in \mathbb{C}$, then $\varphi + \psi$ and $c\varphi$ are again G-linear (check this!), so $\mathrm{Hom}_G(V, W)$ is a complex vector space. For the rest of this problem set G is a finite group and all representations are finite dimensional, so by Problem 6.10 a nonzero representation is irreducible if and only if its only invariant subspaces are $\mathbf{0}$ and itself.

PROBLEM 6.11

Let V and W be G-representations and let $\varphi : V \to W$ be a G-linear map.

1. Prove that $\ker \varphi$ is a G-invariant subspace of V and that $\mathrm{im}\varphi$ is a G-invariant subspace of W.

2. Now suppose V and W are both irreducible. Prove that φ is either the zero map or an isomorphism of G-representations. This result is known as **Schur's lemma**.

3. Suppose V is irreducible and $\varphi : V \to V$ is G-linear. Prove that $\varphi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$, where 1_V is the identity map. Conclude that if V and W are irreducible, then $\dim \mathrm{Hom}_G(V, W)$ is 1 if $V \cong W$ and 0 otherwise. **Hint:** Since $\mathbb{C}$ is algebraically closed, φ has an eigenvalue.

Schur's lemma is simple to prove but remarkably powerful. Here is a first application.

PROBLEM 6.12

Let G be a finite abelian group, let V be a G-representation, and let $h \in G$. Prove that the map $\rho_h : V \to V$ given by $\rho_h(v) := hv$ is G-linear. Conclude that every irreducible representation of G is 1-dimensional.

6.4 · Uniqueness of Irreducible Decomposition

We can now settle the uniqueness question. The idea is that Schur's lemma lets us *count* how many times an irrep W appears inside V by computing the dimension of $\mathrm{Hom}_G(W, V)$, and this dimension does not depend on any choices.

PROBLEM 6.13

1. Let U, U', W be G-representations. Prove that there is an isomorphism of vector spaces

$$\mathrm{Hom}_G(W, U \oplus U') \cong \mathrm{Hom}_G(W, U) \oplus \mathrm{Hom}_G(W, U').$$

   The same argument with the roles reversed shows that $\mathrm{Hom}_G(U \oplus U', W) \cong \mathrm{Hom}_G(U, W) \oplus \mathrm{Hom}_G(U', W)$.

2. Suppose $V \cong W_1 \oplus \cdots \oplus W_\ell$ with each W_i irreducible, and let W be an irreducible representation. Prove that $\dim \mathrm{Hom}_G(W, V)$ and $\dim \mathrm{Hom}_G(V, W)$ both equal the number of indices i with $W_i \cong W$. Conclude that if $V \cong W'_1 \oplus \cdots \oplus W'_m$ is another decomposition into irreducibles, then $m = \ell$ and, after reordering, $W_i \cong W'_i$ for all i.

Given an irreducible representation W, we call

$$m_W(V) := \dim \mathrm{Hom}_G(W, V)$$

the **multiplicity** of W in V. Problem 6.13 says that V is determined up to isomorphism by its multiplicities: if $W_1, \ldots, W_r$ are representatives for the isomorphism classes of irreps appearing in V, then

$$V \cong W_1^{\oplus m_1} \oplus \cdots \oplus W_r^{\oplus m_r}, \qquad m_i = m_{W_i}(V),$$

where $W^{\oplus m}$ denotes the direct sum of m copies of W.

6.5 · Fixed Vectors and Orbits

The simplest irrep is the trivial representation on $\mathbb{C}$, which we denote by $\mathbf{1}$. Let's compute its multiplicity. If V is a G-representation, the **fixed subspace** of V is

$$V^G := \{v \in V : gv = v \text{ for all } g \in G\}.$$

Since G is finite, we may define the **averaging operator** $P : V \to V$ by

$$Pv := \frac{1}{|G|} \sum_{g \in G} gv.$$

Compare this with the averaged inner product of Problem 6.9: in both cases, averaging over the group produces something G-invariant.

PROBLEM 6.14

Let G be a finite group and let V be a G-representation.

1. Prove that V^G is a G-invariant subspace of V, that P is G-linear, that $\mathrm{im} P \subseteq V^G$, and that $Pv = v$ for every $v \in V^G$. Conclude that $\mathrm{im} P = V^G$ and $P^2 = P$. That is, P is a projection onto V^G.

2. Prove that the map $\mathrm{Hom}_G(\mathbf{1}, V) \to V^G$ given by $\varphi \mapsto \varphi(1)$ is an isomorphism of vector spaces. Conclude that $\dim V^G = m_{\mathbf{1}}(V)$ is the multiplicity of the trivial representation in V.

3. Let X be a finite G-set. For each orbit $O \subseteq X$ let $e_O := \sum_{x \in O} e_x \in \mathbb{C}X$. Prove that the vectors e_O, one for each orbit, form a basis for $(\mathbb{C}X)^G$. Hence $\dim(\mathbb{C}X)^G$ is the number of orbits of G on X.

We can squeeze a little more out of the averaging operator by computing its trace. Recall that the **trace** of a square matrix $A = (a_{ij})$ is the sum of its diagonal entries, $\mathrm{tr}(A) := \sum_i a_{ii}$, and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any square matrices A, B of the same size. Consequently $\mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(AQQ^{-1}) = \mathrm{tr}(A)$, so the trace of a linear map $T : V \to V$ may be defined as the trace of its matrix with respect to any basis. Note also that tr is linear: $\mathrm{tr}(cA + c'B) = c\,\mathrm{tr}(A) + c'\,\mathrm{tr}(B)$.

PROBLEM 6.15

1. Let $P : V \to V$ be a linear map with $P^2 = P$. Prove that $V = \mathrm{im} P \oplus \ker P$, in the sense that $\mathrm{im} P$ and $\ker P$ satisfy the internal direct sum criteria, and deduce that $\mathrm{tr}(P) = \dim \mathrm{im} P$. **Hint:** $v = Pv + (v - Pv)$.

2. Let X be a finite G-set and let $g \in G$. Write $X^g := \{x \in X : gx = x\}$ for the set of fixed points of g. Prove that the trace of g acting on $\mathbb{C}X$ equals $|X^g|$. Then prove **Burnside's lemma**: the number of orbits of G on X is
$$\frac{1}{|G|} \sum_{g \in G} |X^g|.$$

The function $g \mapsto \mathrm{tr}(g)$ sending each group element to the trace of its action on V is called the **character** of V. Burnside's lemma is our first glimpse of how much information the character carries; we will study characters more in depth in the coming weeks.

6.6 · The Regular Representation

Recall from the last problem set that the **regular representation** $\mathbb{C}G$ has basis $\{e_h : h \in G\}$ with $ge_h = e_{gh}$. We claimed then that $\mathbb{C}G$ contains a copy of every representation we could ever want. Let's make that precise. Throughout, ε denotes the identity element of G.

PROBLEM 6.16

Let G be a finite group.

1. Let W be any G-representation. Prove that the map $\mathrm{Hom}_G(\mathbb{C}G, W) \to W$ given by $\varphi \mapsto \varphi(e_\varepsilon)$ is an isomorphism of vector spaces.

2. Prove that every irreducible representation W of G occurs in $\mathbb{C}G$ with multiplicity $m_W(\mathbb{C}G) = \dim W$. Use this to show that G has only finitely many irreducible representations up to isomorphism, and that
$$\sum_W (\dim W)^2 = |G|,$$
where the sum runs over the irreducible representations of G up to isomorphism.

3. Let $G = S_3$. Let $\mathbf{1}$ be the trivial representation, let $\mathbb{C}_{\mathrm{sgn}}$ be the **sign representation** on $\mathbb{C}$ given by $\sigma z := \mathrm{sgn}(\sigma) z$, and let V_2 be the plane from the beginning of this problem set, which we showed is irreducible. Prove that $\mathbf{1}$, $\mathbb{C}_{\mathrm{sgn}}$, and V_2 are pairwise non-isomorphic and that every irreducible representation of S_3 is isomorphic to one of them. Conclude that
$$\mathbb{C}S_3 \cong \mathbf{1} \oplus \mathbb{C}_{\mathrm{sgn}} \oplus V_2 \oplus V_2,$$
and find explicit vectors $u, u' \in \mathbb{C}S_3$ such that $\mathrm{Span}(u) \cong \mathbf{1}$ and $\mathrm{Span}(u') \cong \mathbb{C}_{\mathrm{sgn}}$.

The identity $\sum_W (\dim W)^2 = |G|$ is a powerful constraint. For instance, combined with Problem 6.12 it shows that a finite abelian group G has exactly $|G|$ irreducible representations, and conversely that a group with an irreducible representation of dimension greater than 1 cannot be abelian.

6.7 · Challenge: Doubly Transitive Actions

The following is a challenge problem for those who want to push these ideas further. We will use the counting tools developed above to build new irreducible representations out of permutation representations. If X and Y are G-sets, then G acts on $Y \times X$ by $g(y, x) := (gy, gx)$.

PROBLEM 6.17 · (Challenge)

1. Let X and Y be finite G-sets. A linear map $\varphi : \mathbb{C}X \to \mathbb{C}Y$ is determined by the numbers $a_{y,x} \in \mathbb{C}$ defined by $\varphi(e_x) = \sum_{y \in Y} a_{y,x} e_y$. Prove that φ is G-linear if and only if $a_{gy,gx} = a_{y,x}$ for all $g \in G$, $x \in X$, $y \in Y$. Conclude that $\dim \mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}Y)$ is the number of orbits of G on $Y \times X$.

2. Let V and V' be G-representations. Prove that

$$\dim \mathrm{Hom}_G(V, V') = \sum_W m_W(V)\, m_W(V'),$$

where the sum runs over the irreps of G up to isomorphism. Deduce that V is irreducible if and only if $\dim \mathrm{Hom}_G(V, V) = 1$.

3. Let X be a G-set with $|X| \geq 2$. We say G acts **doubly transitively** on X if for any $x \neq y$ and $x' \neq y'$ in X there is some $g \in G$ with $gx = x'$ and $gy = y'$. Prove that the following are equivalent:

   (a) G acts doubly transitively on X;

   (b) G has exactly two orbits on $X \times X$;

   (c) $\mathbb{C}X \cong \mathbf{1} \oplus W$ where W is irreducible and $W \not\cong \mathbf{1}$.

   Show that in this case W may be taken to be $\{\sum_x c_x e_x : \sum_x c_x = 0\}$.

4. Applications.

   (a) Prove that for $n \geq 2$ the **standard representation** $V_{n-1} := \{(c_1, \ldots, c_n) \in \mathbb{C}^n : \sum_i c_i = 0\}$ of S_n is irreducible. (For $n = 3$ this is the plane V_2 from the beginning of this problem set.)

   (b) Prove that for $n \geq 4$, V_{n-1} remains irreducible when regarded as a representation of the alternating group A_n. What goes wrong when $n = 3$?

   (c) Let p be a prime and let $G = \mathrm{Aff}(\mathbb{F}_p)$ be the group of functions $\mathbb{F}_p \to \mathbb{F}_p$ of the form $x \mapsto ax + b$ with $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$, under composition. Determine all irreducible representations of G up to isomorphism: prove that there are exactly p of them, namely $p - 1$ of dimension 1 and one of dimension $p - 1$. **Hint:** Look at the map $G \to \mathbb{F}_p^\times$ sending $x \mapsto ax + b$ to a, and recall the remark following Problem 6.16. (When $p = 3$, check that $G \cong S_3$ and you recover Problem 6.16(3).)

   (d) Let $n \geq 4$ and let X be the set of 2-element subsets of $[n]$, with the natural action of S_n. Prove that $\mathbb{C}X \cong \mathbf{1} \oplus V_{n-1} \oplus U$ where U is an irreducible representation of S_n of dimension $n(n-3)/2$, not isomorphic to $\mathbf{1}$ or V_{n-1}. In particular, this produces a 2-dimensional irrep of S_4 and a 5-dimensional irrep of S_5. **Hint:** Use part (1) to compute $\dim \mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}X)$ and $\dim \mathrm{Hom}_G(\mathbb{C}[n], \mathbb{C}X)$.