MATH 301 · FALL 2026

Problem Set 6: Sum Decompositions

SOLUTIONS

First, there's the room you can see through the glass—that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair—all but the bit behind the fireplace. Oh! I do so wish I could see that *bit!*

From *Through the Looking Glass* by Lewis Carroll

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

Let's recall a definition from the end of the last problem set.

DEFINITION 6.1
Let G be a group and let V be a G-representation. A subspace $U \subseteq V$ is called a **G-invariant subspace** or a **G-subrepresentation** if $gU \subseteq U$ for all $g \in G$.

In Problem 5.5 we analyzed the S_3-invariant subspaces of $V := \mathbb{C}[3] \cong \mathbb{C}^3$ where $[3] = \{1, 2, 3\}$ is an S_3-set. There are precisely 4 of them:

1. $\mathbf{0} := \{0\}$,
2. $V_1 := \text{Span}((1, 1, 1))$,
3. $V_2 := \text{Span}((1, -1, 0), (1, 0, -1))$,
4. V,

where we use $\text{Span}(v_1, \ldots, v_k)$ to denote the span of the v_i. The $\mathbf{0}$ and V cases are boring, every representation V has $\mathbf{0}$ and V as invariant subspaces. The V_1 and V_2 are more interesting. For one, notice that they are orthogonal to one another.

The fact that these spaces are orthogonal to one another implies that the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, -1, 0)$, and $v_3 = (1, 0, -1)$ form a basis $\mathfrak{B}$ for V. This is a special basis for our group action. If we express the elements of S_3 in terms of the basis $\mathfrak{B}$ they have a special form. For example, consider the action of (123). To find the matrix representation $[(123)]_{\mathfrak{B}}$ with respect to the basis $\mathfrak{B}$ note that

$$\begin{aligned}(123)v_1 &= (123)(1, 1, 1) = (1, 1, 1) = v_1\\ (123)v_2 &= (123)(1, -1, 0) = (0, 1, -1) = v_3 - v_2\\ (123)v_3 &= (123)(1, 0, -1) = (-1, 1, 0) = -v_2.\end{aligned}$$

Hence

$$[(123)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

I'll let you work out the rest.

PROBLEM 6.1

Work out the matrix representation $[g]_{\mathfrak{B}}$ for each $g \in S_3$. What do you notice about the form of these matrices?

SOLUTION

Recall that $\sigma \in S_3$ acts on $\mathbb{C}^3$ by permuting coordinates, $\sigma e_i = e_{\sigma(i)}$. Every σ fixes $v_1 = e_1 + e_2 + e_3$. Since $v_2 = e_1 - e_2$ and $v_3 = e_1 - e_3$, we have $\sigma v_2 = e_{\sigma(1)} - e_{\sigma(2)}$ and $\sigma v_3 = e_{\sigma(1)} - e_{\sigma(3)}$, and each difference $e_i - e_j$ is a combination of v_2 and v_3:

$$e_1 - e_2 = v_2, \qquad e_1 - e_3 = v_3, \qquad e_2 - e_3 = v_3 - v_2,$$

together with the negatives of these. Writing the coordinates of σv_j in the jth column, we find

$$[e]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [(12)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad [(13)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix},$$

$$[(23)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [(123)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad [(132)]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

For instance, $(12)v_2 = e_2 - e_1 = -v_2$ and $(12)v_3 = e_2 - e_3 = v_3 - v_2$, which give the second and third columns of $[(12)]_{\mathfrak{B}}$. Every one of these matrices is *block diagonal*,

$$[g]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 \\ 0 & A_g \end{bmatrix},$$

with a 1×1 block, a 2×2 block A_g, and zeros elsewhere. The 1×1 block records that g acts trivially on the line $V_1 = \mathrm{Span}(v_1)$, and A_g is the matrix of g acting on the plane $V_2 = \mathrm{Span}(v_2, v_3)$ with respect to the basis v_2, v_3. The off-diagonal blocks vanish precisely because V_1 and V_2 are both S_3-invariant: gv_1 involves only v_1, and gv_2, gv_3 involve only v_2 and v_3. One can also check that $\det A_g = \mathrm{sgn}(g)$, and that A_g has trace 0 when g is a transposition and trace -1 when g is a 3-cycle, as expected for reflections and 120° rotations of the plane V_2. □

The special form the matrix representations of S_3 take with respect to the basis $\mathfrak{B}$ is a reflection of the structure of V as an S_3-representation. To understand this we need to review the notion of a *direct sum* of vector spaces.

6.1 · Direct Sums

If U and V are vector spaces, then their **direct sum** $U \oplus V$ is simply the Cartesian product $U \times V$ with the vector space operations defined componentwise.[1] We typically write $u \oplus v$ to denote (u, v) in order to remind ourselves to think of $u \oplus v$ as a formal sum of two vectors. Note that the vector spaces U and V could be totally unrelated so that there is no natural way to add vectors between U and V. The direct sum allows us to formally combine these into one vector space where we use the special symbol $\oplus$ to denote the formal sum combining these disparate vectors.

If U and V are G-representations, say with actions $\rho : G \to \mathrm{Aut}(U)$ and $\rho' : G \to \mathrm{Aut}(V)$, then $U \oplus V$ also forms a representation of G with action $\rho \oplus \rho' : G \to \mathrm{Aut}(U \oplus V)$ defined by

$$(\rho \oplus \rho')_g(u \oplus v) := \rho_g(u) \oplus \rho'_g(v).$$

Or with the actions left implicit, this simplifies to

$$g(u \oplus v) = gu \oplus gv.$$

Hence G acts separately on U and V with no interaction between them.

Before going further, let's say what it means for two G-representations to be isomorphic. The definition follows the same pattern as G-maps of G-sets and R-linear maps of R-modules: the map should commute with the action.

DEFINITION 6.2

Let U and V be G-representations. A linear map $\varphi : U \to V$ is called **G-linear** (or a **G-map**) if

$$\varphi(gu) = g\varphi(u)$$

for all $g \in G$ and $u \in U$. A **G-linear isomorphism** is a bijective G-linear map. If a G-linear isomorphism $U \to V$ exists we say U and V are **isomorphic as G-representations** and write $U \cong V$.

Just as for R-modules, the inverse of a bijective G-linear map is automatically G-linear, so a G-linear isomorphism has a G-linear inverse.

PROBLEM 6.2

1. Let U, V be vector spaces. Suppose that $\mathfrak{B}_U := \{u_1, \ldots, u_m\}$ is a basis for U and $\mathfrak{B}_V := \{v_1, \ldots, v_n\}$ is a basis for V. Prove that

   $$\mathfrak{B}_{U \oplus V} := \{u_1 \oplus 0, \ldots, u_m \oplus 0, 0 \oplus v_1, \ldots, 0 \oplus v_n\}$$

   is a basis for $U \oplus V$.

[1] Important caveat: There is an important difference between Cartesian products and direct sums of *infinitely many* vector spaces, but for *finitely many* vector spaces, the two concepts are the same.

SOLUTION

Let $x \oplus y \in U \oplus V$. Since $\mathfrak{B}_U$ and $\mathfrak{B}_V$ are bases, we may write $x = \sum_i a_i u_i$ and $y = \sum_j b_j v_j$ for some scalars $a_i, b_j \in \mathbb{C}$. Since the vector space operations in $U \oplus V$ are defined componentwise,

$$x \oplus y = \left(\sum_i a_i u_i\right) \oplus \left(\sum_j b_j v_j\right) = \sum_i a_i (u_i \oplus 0) + \sum_j b_j (0 \oplus v_j),$$

so $\mathfrak{B}_{U\oplus V}$ spans $U \oplus V$. For linear independence, suppose $\sum_i a_i (u_i \oplus 0) + \sum_j b_j (0 \oplus v_j) = 0 \oplus 0$. The left side equals $(\sum_i a_i u_i) \oplus (\sum_j b_j v_j)$, so comparing components gives $\sum_i a_i u_i = 0$ in U and $\sum_j b_j v_j = 0$ in V. Since $\mathfrak{B}_U$ and $\mathfrak{B}_V$ are linearly independent, all of the a_i and b_j are zero. Hence $\mathfrak{B}_{U\oplus V}$ is a basis, and in particular $\dim(U \oplus V) = \dim U + \dim V$. □

2. Now suppose that U and V are both G-representations. Given $g \in G$, let $[g]_U$ and $[g]_V$ denote the matrix representations for g acting on U and V with respect to the bases $\mathfrak{B}_U$ and $\mathfrak{B}_V$, respectively. Similarly, let $[g]_{U\oplus V}$ denote the matrix representation of g with respect to $\mathfrak{B}_{U\oplus V}$. Prove that

$$[g]_{U\oplus V} = \begin{bmatrix} [g]_U & 0 \\ 0 & [g]_V \end{bmatrix},$$

where the 0 here represent rectangular matrices of zeros. Matrices in this form are called **block diagonal** matrices.

SOLUTION

Write $[g]_U = (a_{ij})$ and $[g]_V = (b_{ij})$. By definition of the matrix of a linear map with respect to a basis, this means

$$gu_j = \sum_{i=1}^{m} a_{ij} u_i \qquad \text{and} \qquad gv_j = \sum_{i=1}^{n} b_{ij} v_i.$$

The jth column of $[g]_{U\oplus V}$ records the coordinates of g applied to the jth vector of $\mathfrak{B}_{U\oplus V}$. For $1 \le j \le m$,

$$g(u_j \oplus 0) = gu_j \oplus g0 = \left(\sum_{i=1}^{m} a_{ij} u_i\right) \oplus 0 = \sum_{i=1}^{m} a_{ij} (u_i \oplus 0),$$

so the jth column of $[g]_{U\oplus V}$ is $(a_{1j}, \ldots, a_{mj}, 0, \ldots, 0)$: the jth column of $[g]_U$ followed by n zeros. Similarly, for $1 \le j \le n$,

$$g(0 \oplus v_j) = 0 \oplus gv_j = \sum_{i=1}^{n} b_{ij} (0 \oplus v_i),$$

so the $(m + j)$th column of $[g]_{U\oplus V}$ is $(0, \ldots, 0, b_{1j}, \ldots, b_{nj})$: m zeros followed by the jth column of $[g]_V$. Assembling these columns gives exactly the block diagonal matrix

$$[g]_{U\oplus V} = \begin{bmatrix} [g]_U & 0 \\ 0 & [g]_V \end{bmatrix}.$$ □

While direct sums may be used to combine G-representations, we more often use them to decompose a given G-representation. Let W be a vector space and suppose that $U, V \subseteq W$ are subspaces. We define the **internal sum** $U + V \subseteq W$ to be the subspace

$$U + V = \{u + v : u \in U, v \in V\}.$$

Note that since U and V live inside a common vector space W, it does make sense to add vectors from U and V.

PROBLEM 6.3

Suppose that $U, V \subseteq W$ are subspaces such that $U + V = W$ and $U \cap V = \mathbf{0}$. Prove that $W \cong U \oplus V$. These conditions are called the **internal direct sum criteria**.

SOLUTION

Define $\psi : U \oplus V \to W$ by $\psi(u \oplus v) := u + v$. This makes sense because U and V are both subspaces of the same vector space W. It is linear since the operations on $U \oplus V$ are componentwise:

$$\begin{aligned}\psi\big(c(u \oplus v) + c'(u' \oplus v')\big) &= \psi\big((cu + c'u') \oplus (cv + c'v')\big) \\ &= (cu + c'u') + (cv + c'v') = c\psi(u \oplus v) + c'\psi(u' \oplus v').\end{aligned}$$

The image of ψ is $\{u + v : u \in U, v \in V\} = U + V = W$, so ψ is surjective. If $\psi(u \oplus v) = 0$, then $u = -v$ lies in both U and V, so $u \in U \cap V = \mathbf{0}$. Thus $u = 0$ and $v = -u = 0$, so $\ker \psi = \mathbf{0}$ and ψ is injective. Hence ψ is an isomorphism $U \oplus V \cong W$. □

If W is a G-representation and $U, V \subseteq W$ are subrepresentations satisfying the internal direct sum criteria, then we also have $W \cong U \oplus V$ as G-representations. When studying a representation of G, we often try to decompose it as a sum of smaller, simpler representations. This is the linear representation analog of studying a group action by decomposing the action into orbits.

PROBLEM 6.4

Let X and Y be G-sets. Recall that $X + Y$ is the disjoint union of X and Y. Prove that $\mathbb{C}(X + Y) \cong \mathbb{C}X \oplus \mathbb{C}Y$. **Hint:** Without loss of generality you may assume that $X, Y \subseteq X + Y$. This will allow you to use the internal direct sum criteria.

SOLUTION

Write $Z := X + Y$ and regard X and Y as subsets of Z, as the hint suggests. Then X and Y are G-invariant subsets of Z (the action of G on Z restricts to the given actions on X and Y), and $X \cap Y = \emptyset$.

By definition $\mathbb{C}Z$ has basis $\{e_z : z \in Z\}$. Inside $\mathbb{C}Z$, the vectors e_x with $x \in X$ span a subspace which is exactly $\mathbb{C}X$ (it consists of the formal linear combinations of the e_x), and likewise the e_y with $y \in Y$ span $\mathbb{C}Y$. Both are G-invariant subspaces of $\mathbb{C}Z$: for $x \in X$ we have $ge_x = e_{gx}$ with

SOLUTION (CONTINUED)

$gx \in X$, so g carries the spanning set of $\mathbb{C}X$ into $\mathbb{C}X$, and similarly for $\mathbb{C}Y$.
We check the internal direct sum criteria for $\mathbb{C}X, \mathbb{C}Y \subseteq \mathbb{C}Z$. Every basis vector e_z of $\mathbb{C}Z$ lies in $\mathbb{C}X$ or in $\mathbb{C}Y$ according to whether $z \in X$ or $z \in Y$, so $\mathbb{C}X + \mathbb{C}Y$ contains a basis of $\mathbb{C}Z$ and hence $\mathbb{C}X + \mathbb{C}Y = \mathbb{C}Z$. If $v \in \mathbb{C}X \cap \mathbb{C}Y$, then writing v in the basis $\{e_z\}$, membership in $\mathbb{C}X$ says that only coefficients indexed by X can be nonzero, while membership in $\mathbb{C}Y$ says that only coefficients indexed by Y can be nonzero. Since $X \cap Y = \emptyset$ and coefficients with respect to a basis are unique, every coefficient of v is zero, so $v = 0$ and $\mathbb{C}X \cap \mathbb{C}Y = \mathbf{0}$.
By Problem 6.3 and the remark following it, the subrepresentations $\mathbb{C}X$ and $\mathbb{C}Y$ satisfy the internal direct sum criteria, so $\mathbb{C}(X+Y) = \mathbb{C}Z \cong \mathbb{C}X \oplus \mathbb{C}Y$ as G-representations. □

As we recursively decompose a representation, eventually we reach invariant subspaces which cannot decompose any further. These are called *irreducible representations* or *irreps* for short.

DEFINITION 6.3
A nonzero G-representation W is said to be **irreducible** if $W \ncong U \oplus V$ for any U, V with positive dimension.

For example, the plane $V_2 = \mathrm{Span}((1,-1,0),(1,0,-1))$ is an irreducible S_3-representation: If not, then $V_2 \cong U_1 \oplus U_2$ where the U_i both have dimension 1. That would mean V_2 contains two S_3-invariant lines. However note that (123) acts by a 120° rotation of the plane, hence does not leave any lines invariant. Thus no U_1 and U_2 exist.

Just as every G-set breaks down into a disjoint union of transitive G-sets, every representation breaks down into a direct sum of irreducibles.

PROBLEM 6.5

Prove that every nonzero finite dimensional representation V is isomorphic to a finite direct sum of irreducible representations.

SOLUTION

We induct on $\dim V$. If $\dim V = 1$, then V is irreducible: if $V \cong U \oplus U'$, then $\dim U + \dim U' = \dim(U \oplus U') = 1$ by Problem 6.2, so U and U' cannot both have positive dimension.
Now let $\dim V = n > 1$ and suppose the result holds for all representations of dimension less than n. If V is irreducible, we are done. Otherwise $V \cong U \oplus U'$ for some G-representations U, U' of positive dimension. Since $\dim U + \dim U' = n$, both U and U' have dimension strictly less than n, so by induction

$$U \cong W_1 \oplus \cdots \oplus W_k \qquad \text{and} \qquad U' \cong W_{k+1} \oplus \cdots \oplus W_\ell$$

for some irreducible representations W_i. If $\alpha : U \to U_0$ and $\beta : U' \to U_0'$ are isomorphisms of G-representations, then so is $\alpha \oplus \beta : U \oplus U' \to U_0 \oplus U_0'$, $u \oplus u' \mapsto \alpha(u) \oplus \beta(u')$: it is linear, it

SOLUTION (CONTINUED)

commutes with G since $\alpha(gu) \oplus \beta(gu') = g\alpha(u) \oplus g\beta(u')$, and it has inverse $\alpha^{-1} \oplus \beta^{-1}$. Hence

$$V \cong U \oplus U' \cong (W_1 \oplus \cdots \oplus W_k) \oplus (W_{k+1} \oplus \cdots \oplus W_\ell) \cong W_1 \oplus \cdots \oplus W_\ell,$$

where the last isomorphism simply regroups the components: direct sums are associative up to the evident isomorphism $(A \oplus B) \oplus C \cong A \oplus (B \oplus C)$, which also commutes with G. This completes the induction. □

We will ultimately show that every finite dimensional representation has an essentially unique decomposition into a direct sum of irreps. But first some preliminary work. For instance, how do we know whether a given representation W is irreducible? If $W \cong U \oplus V$, that implies W has proper invariant subspaces isomorphic to U and V, so I could start by looking for an invariant subspace U. Say we find an invariant subspace U: will there always exist a complementary invariant subspace V such that $W \cong U \oplus V$? We call an invariant subspace $V \subseteq W$ such that $U + V = W$ and $U \cap V = \mathbf{0}$ a **complementary subspace** of U.

PROBLEM 6.6

Let $C_\infty = \langle g \rangle$ be the infinite cyclic group. An n-dimensional representation of C_∞ is equivalent to the choice of an invertible $n \times n$ matrix. For example, consider the 2-dimensional representation $W = \mathbb{C}^2$ of C_∞ given by

$$g \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Prove that $U = \mathrm{Span}(e_1)$ is a C_∞-invariant subspace and that U has no complementary invariant subspace.

SOLUTION

Let $A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that g^n acts on $W = \mathbb{C}^2$ by $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for every $n \in \mathbb{Z}$. Since $Ae_1 = e_1$, we have $g^n e_1 = e_1$ for all n, so $U = \mathrm{Span}(e_1)$ is C_∞-invariant (indeed, C_∞ acts trivially on U).
Suppose $V \subseteq W$ were a complementary invariant subspace, so that $U + V = W$ and $U \cap V = \mathbf{0}$. By Problems 6.2 and 6.3, $\dim W = \dim U + \dim V$, so $\dim V = 1$, say $V = \mathrm{Span}(v)$ with $v \neq 0$. Invariance of V under g means $Av \in \mathrm{Span}(v)$, that is, $Av = \lambda v$ for some $\lambda \in \mathbb{C}$: the vector v is an eigenvector of A. But the characteristic polynomial of A is $(t-1)^2$, so the only eigenvalue is 1, and

$$\ker(A - I) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Span}(e_1) = U.$$

Hence $v \in U$, so $V = U$, contradicting $U \cap V = \mathbf{0}$. Therefore U has no complementary invariant subspace. □

This problem shows that complementary invariant subspaces do not always exist. However, the issue in this example stems entirely from the fact that C_∞ is an infinite group. The situation with finite groups works out quite neatly. To explain, we need to first discuss inner products.

6.2 · Inner Products

Given two vectors $v = (a_1, \ldots, a_n), w = (b_1, \ldots, b_n) \in \mathbb{R}^n$, their dot product is

$$v \cdot w := \sum_{i=1}^{n} a_i b_i.$$

The dot product is used to measure the correlation between two vectors. This can be seen from the formula

$$v \cdot w = |v||w| \cos\theta,$$

where $|v|$ is the magnitude of v and θ is the angle between v and w. The idea is that if the angle between v and w is small, then $\cos(\theta)$ is close to one and the dot product is essentially just the product of their magnitudes. But if v and w are close to being orthogonal, then $\cos(\theta)$ is nearly zero.

The familiar dot product generalizes to the notion of an *inner product*.

DEFINITION 6.4

Let V be a $\mathbb{C}$-vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called an **inner product** if

1. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$,
2. $\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$ for all $v_1, v_2, w \in V$ and all $c_1, c_2 \in \mathbb{C}$,
3. $\langle v, v \rangle > 0$ for all nonzero $v \in V$.

A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called an **inner product space**.

PROBLEM 6.7

Let's review/get to know inner products. Suppose V is an inner product space.

1. Prove that $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$ (hence that the third axiom defining the inner product makes sense).

   SOLUTION

   Taking $w = v$ in the first axiom gives $\langle v, v \rangle = \overline{\langle v, v \rangle}$. A complex number equal to its own conjugate is real, so $\langle v, v \rangle \in \mathbb{R}$ and the inequality $\langle v, v \rangle > 0$ in the third axiom makes sense. □

2. Prove that $\langle v, c_1 w_1 + c_2 w_2 \rangle = \overline{c_1} \langle v, w_1 \rangle + \overline{c_2} \langle v, w_2 \rangle$ for all $v, w_1, w_2 \in V$ and all $c_1, c_2 \in \mathbb{C}$.

SOLUTION

Using the first axiom, then linearity in the first slot, then the fact that complex conjugation respects sums and products,

$$\begin{aligned}\langle v, c_1 w_1 + c_2 w_2\rangle &= \overline{\langle c_1 w_1 + c_2 w_2, v\rangle} = \overline{c_1\langle w_1, v\rangle + c_2\langle w_2, v\rangle} \\ &= \overline{c_1}\,\overline{\langle w_1, v\rangle} + \overline{c_2}\,\overline{\langle w_2, v\rangle} = \overline{c_1}\langle v, w_1\rangle + \overline{c_2}\langle v, w_2\rangle,\end{aligned}$$

where the last step uses the first axiom once more. So an inner product is linear in the first slot and *conjugate-linear* in the second. □

3. Prove that $\langle 0, w\rangle = \langle v, 0\rangle = 0$ for all $v, w \in V$.

SOLUTION

Since the zero vector satisfies $0 = 0 \cdot 0$ (the scalar 0 times the vector 0), linearity in the first slot gives

$$\langle 0, w\rangle = \langle 0 \cdot 0, w\rangle = 0 \cdot \langle 0, w\rangle = 0.$$

Then $\langle v, 0\rangle = \overline{\langle 0, v\rangle} = \overline{0} = 0$. □

While we cannot use a general inner product to measure angles in a complex vector space, we can still use them to detect when two vectors are orthogonal to one another: we say $v, w \in V$ are **orthogonal** if $\langle v, w\rangle = 0$.

If V is a vector space and $U \subseteq V$ is a subspace, then in general there is no canonical complementary subspace $W \subseteq V$ such that $V \cong U \oplus W$. You can always find a complement, but it will require making some choices. However, one thing an inner product gives us is canonical complements.

PROBLEM 6.8

1. Let V be a finite dimensional inner product space and let $U \subseteq V$ be a subspace. Prove that

$$U^\perp := \{v \in V : \langle u, v\rangle = 0 \text{ for all } u \in U\}$$

is a subspace of V and that $V \cong U \oplus U^\perp$. **Hint:** Choose a basis $u_1, \ldots, u_k$ of U and consider the linear transformation $T : V \to \mathbb{C}^k$ defined by

$$T(v) := (\langle v, u_1\rangle, \ldots, \langle v, u_k\rangle).$$

We call $U^\perp$ the **orthogonal complement** of U.

SOLUTION

Let $n := \dim V$ and $k := \dim U$.
$U^\perp$ is a subspace. By Problem 6.7(3), $\langle u, 0\rangle = 0$ for all $u \in U$, so $0 \in U^\perp$. If $v, v' \in U^\perp$ and $c, c' \in \mathbb{C}$, then for every $u \in U$ conjugate-linearity in the second slot (Problem 6.7(2)) gives

$$\langle u, cv + c'v'\rangle = \overline{c}\langle u, v\rangle + \overline{c'}\langle u, v'\rangle = 0,$$

so $cv + c'v' \in U^\perp$.
$U \cap U^\perp = \mathbf{0}$. If $u \in U \cap U^\perp$, then $\langle u, u\rangle = 0$, which by the third axiom forces $u = 0$.
$U + U^\perp = V$. Choose a basis $u_1, \ldots, u_k$ of U and define $T : V \to \mathbb{C}^k$ by

$$T(v) := \big(\langle v, u_1\rangle, \ldots, \langle v, u_k\rangle\big).$$

Each coordinate is linear in v by the second axiom, so T is linear. A vector v lies in $\ker T$ exactly when $\langle v, u_i\rangle = 0$ for all i. Since every $u \in U$ is a linear combination of the u_i, this is equivalent to $\langle v, u\rangle = 0$ for all $u \in U$, which after conjugating is the condition $v \in U^\perp$. Thus $\ker T = U^\perp$, and rank–nullity gives $\dim U^\perp = n - \operatorname{rank} T \geq n - k$. Applying Problem 6.3 to the subspaces U and $U^\perp$ of $U + U^\perp$ gives $U + U^\perp \cong U \oplus U^\perp$, so by Problem 6.2

$$\dim(U + U^\perp) = \dim U + \dim U^\perp \geq k + (n - k) = n = \dim V.$$

Hence $U + U^\perp = V$ (and in fact $\dim U^\perp = n - k$).
The internal direct sum criteria now give $V \cong U \oplus U^\perp$. □

2. Now suppose that V is a G-representation and that $U \subseteq V$ is an invariant subspace. Furthermore, let's suppose that V has an inner product $\langle \cdot, \cdot \rangle$ which is G-invariant:

$$\langle gv, gw\rangle = \langle v, w\rangle$$

for all $g \in G$ and $v, w \in V$. Prove that $U^\perp$ is also G-invariant and that $V \cong U \oplus U^\perp$ as G-representations.

SOLUTION

Let $v \in U^\perp$ and $g \in G$. For any $u \in U$, invariance of the inner product applied with the group element g^{-1} gives

$$\langle u, gv\rangle = \langle g^{-1}u, g^{-1}gv\rangle = \langle g^{-1}u, v\rangle = 0,$$

since $g^{-1}u \in U$ by invariance of U and $v \in U^\perp$. Hence $gv \in U^\perp$, so $U^\perp$ is G-invariant. By part (1), U and $U^\perp$ satisfy the internal direct sum criteria, and both are subrepresentations, so by Problem 6.3 (and the remark following it) the map $u \oplus v \mapsto u + v$ is an isomorphism of G-representations $U \oplus U^\perp \cong V$. □

3. Prove that the C_∞-representation W constructed in Problem 6.6 does not have any C_∞-invariant inner product.

SOLUTION

Suppose for contradiction that W had a C_∞-invariant inner product. The subspace $U = \mathrm{Span}(e_1)$ is C_∞-invariant by Problem 6.6, so by part (2) its orthogonal complement $U^\perp$ is a C_∞-invariant subspace with $U + U^\perp = W$ and $U \cap U^\perp = \mathbf{0}$. That is, $U^\perp$ is a complementary invariant subspace of U, which Problem 6.6 shows does not exist. □

Hence if V is a G-representation with a G-invariant inner product, then every subrepresentation has a complement. Thus a natural question to ask is: which representations have a G-invariant inner product? Suppose G is a finite group and that V is a G-representation with an inner product $\langle \cdot, \cdot \rangle$ (not necessarily G-invariant). We then define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle.$$

PROBLEM 6.9

Prove that $\langle \cdot, \cdot \rangle_G$ is a G-invariant inner product on V.

SOLUTION

The sum defining $\langle v, w \rangle_G$ is finite since G is finite, so $\langle v, w \rangle_G$ is a well-defined complex number. We check the three axioms.

Conjugate symmetry. Since complex conjugation respects sums and $1/|G|$ is real,

$$\overline{\langle w, v \rangle_G} = \frac{1}{|G|} \sum_{g \in G} \overline{\langle gw, gv \rangle} = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle = \langle v, w \rangle_G.$$

Linearity in the first slot. Each g acts linearly, so $g(c_1 v_1 + c_2 v_2) = c_1 g v_1 + c_2 g v_2$, and then linearity of $\langle \cdot, \cdot \rangle$ in the first slot gives

$$\langle c_1 v_1 + c_2 v_2, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \big(c_1 \langle gv_1, gw \rangle + c_2 \langle gv_2, gw \rangle \big) = c_1 \langle v_1, w \rangle_G + c_2 \langle v_2, w \rangle_G.$$

Positivity. If $v \neq 0$, then $gv \neq 0$ for every $g \in G$, since g acts by an invertible linear map: if $gv = 0$ then $v = g^{-1}(gv) = 0$. Hence each term $\langle gv, gv \rangle$ is a positive real number, and so is their average $\langle v, v \rangle_G$.

G-invariance. For $h \in G$,

$$\langle hv, hw \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle ghv, ghw \rangle = \frac{1}{|G|} \sum_{g' \in G} \langle g'v, g'w \rangle = \langle v, w \rangle_G,$$

where we substituted $g' = gh$: as g runs over G so does $g' = gh$, because right multiplication by h is a bijection $G \to G$ (with inverse right multiplication by h^{-1}). Thus $\langle \cdot, \cdot \rangle_G$ is a G-invariant inner product on V. □

Thus if G is finite we can take any inner product on V and average it over the group action in order to construct a G-invariant inner product. Notice that this only works if G is finite, since if G is infinite we run into convergence issues with the infinite sum and there's the issue of dividing by $|G| = \infty$. [2]

There's one last detail to check: Which G-representations have an inner product? We need one in order to average it and get a G-invariant inner product.

PROBLEM 6.10

1. Construct an explicit inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ for all $n \geq 1$.

SOLUTION

For $v = (a_1, \ldots, a_n)$ and $w = (b_1, \ldots, b_n)$ in $\mathbb{C}^n$ define

$$\langle v, w \rangle := \sum_{i=1}^{n} a_i \overline{b_i}.$$

This is the **standard inner product** on $\mathbb{C}^n$; when all coordinates are real it is the usual dot product. We check the axioms.
Conjugate symmetry. $\overline{\langle w, v \rangle} = \overline{\sum_i b_i \overline{a_i}} = \sum_i \overline{b_i}\, a_i = \langle v, w \rangle$.
Linearity in the first slot. If $v' = (a'_1, \ldots, a'_n)$ and $c, c' \in \mathbb{C}$, then $\langle cv + c'v', w \rangle = \sum_i (ca_i + c'a'_i)\overline{b_i} = c\langle v, w \rangle + c'\langle v', w \rangle$.
Positivity. $\langle v, v \rangle = \sum_i a_i \overline{a_i} = \sum_i |a_i|^2$ is a sum of nonnegative reals, and it is zero only if every $|a_i| = 0$, that is, only if $v = 0$. □

2. Let V be an n-dimensional complex vector space and suppose that $\varphi : V \to \mathbb{C}^n$ is an isomorphism. Prove that the function

$$\langle v, w \rangle_\varphi := \langle \varphi(v), \varphi(w) \rangle$$

defines an inner product on V.

SOLUTION

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^n$ from part (1). Conjugate symmetry and linearity in the first slot are inherited from $\mathbb{C}^n$ because φ is linear:

$$\langle v, w \rangle_\varphi = \langle \varphi(v), \varphi(w) \rangle = \overline{\langle \varphi(w), \varphi(v) \rangle} = \overline{\langle w, v \rangle_\varphi}$$

[2] We can sometimes make this work when G is infinite, but we need a topology on both G and V. When the group G is compact (like the circle group $U(1)$) we can make this work using an integral in place of the sum. Essentially everything we prove about representations of finite groups may be generalized to compact topological groups as well.

SOLUTION (CONTINUED)

and

$$\begin{aligned}\langle c_1v_1 + c_2v_2, w\rangle_\varphi &= \langle c_1\varphi(v_1) + c_2\varphi(v_2), \varphi(w)\rangle \\ &= c_1\langle\varphi(v_1), \varphi(w)\rangle + c_2\langle\varphi(v_2), \varphi(w)\rangle = c_1\langle v_1, w\rangle_\varphi + c_2\langle v_2, w\rangle_\varphi.\end{aligned}$$

For positivity, if $v \neq 0$ then $\varphi(v) \neq 0$ since φ is injective, so $\langle v, v\rangle_\varphi = \langle\varphi(v), \varphi(v)\rangle > 0$. (Only the injectivity of φ was used; surjectivity is not needed.) Since every n-dimensional vector space admits an isomorphism $\varphi : V \to \mathbb{C}^n$, obtained by choosing a basis, every finite dimensional vector space has an inner product. □

3. If G is a finite group and V is a nonzero finite dimensional G-representation, prove that V is irreducible if and only if the only G-invariant subspaces of V are $\mathbf{0}$ and V itself.

SOLUTION

Suppose first that V has an invariant subspace U with $U \neq \mathbf{0}$ and $U \neq V$. By parts (1) and (2), V has an inner product, and by Problem 6.9 averaging it over G produces a G-invariant inner product on V. By Problem 6.8(2), $V \cong U \oplus U^\perp$ as G-representations. Here $\dim U > 0$ since $U \neq \mathbf{0}$, and $\dim U^\perp = \dim V - \dim U > 0$ since $U \neq V$. Thus V is not irreducible.
Conversely, suppose V is not irreducible, so there is an isomorphism of G-representations $\psi : U \oplus U' \to V$ with U and U' of positive dimension. Let $U_0 := \psi(U \oplus 0) \subseteq V$ be the image of the subspace $U \oplus 0 := \{u \oplus 0 : u \in U\}$. Since ψ is linear, U_0 is a subspace, and it is G-invariant because ψ commutes with G: for $u \in U$,

$$g\psi(u \oplus 0) = \psi(g(u \oplus 0)) = \psi(gu \oplus 0) \in U_0.$$

Since ψ is injective, $\dim U_0 = \dim U > 0$, so $U_0 \neq \mathbf{0}$; and $\dim U_0 = \dim V - \dim U' < \dim V$, so $U_0 \neq V$. Thus V has an invariant subspace other than $\mathbf{0}$ and V.
Combining the two directions, V is irreducible if and only if its only invariant subspaces are $\mathbf{0}$ and V. Note that the second direction did not use that G is finite; finiteness (through the averaging trick) is needed only for the first. □

Thus, to summarize what we have now shown: If G is a finite group and V is a finite dimensional G-representation, then V has a G-invariant inner product. The irreducible representations of G are precisely those representations V whose only invariant subspaces are $\mathbf{0}$ and V itself. Every finite dimensional representation V of G may be decomposed as a sum of irreps.

This raises two natural questions. First, the proof of Problem 6.5 involved choices; is the resulting list of irreps unique? Second, how do we find the irreps of a given group, and how do we recognize them when we see them? The key to both questions is to study the G-linear maps between representations.

6.3 · Schur's Lemma

Let V and W be G-representations. We write $\mathrm{Hom}_G(V,W)$ for the set of all G-linear maps $\varphi : V \to W$. If $\varphi, \psi \in \mathrm{Hom}_G(V,W)$ and $c \in \mathbb{C}$, then $\varphi + \psi$ and $c\varphi$ are again G-linear (check this!), so $\mathrm{Hom}_G(V,W)$ is a complex vector space. For the rest of this problem set G is a finite group and all representations are finite dimensional, so by Problem 6.10 a nonzero representation is irreducible if and only if its only invariant subspaces are $\mathbf{0}$ and itself.

PROBLEM 6.11

Let V and W be G-representations and let $\varphi : V \to W$ be a G-linear map.

1. Prove that $\ker\varphi$ is a G-invariant subspace of V and that $\mathrm{im}\varphi$ is a G-invariant subspace of W.

SOLUTION

Both are subspaces by linear algebra, so we only need to check invariance. If $v \in \ker\varphi$ and $g \in G$, then $\varphi(gv) = g\varphi(v) = g0 = 0$, so $gv \in \ker\varphi$. If $w \in \mathrm{im}\varphi$, say $w = \varphi(v)$, then $gw = g\varphi(v) = \varphi(gv) \in \mathrm{im}\varphi$. □

2. Now suppose V and W are both irreducible. Prove that φ is either the zero map or an isomorphism of G-representations. This result is known as **Schur's lemma**.

SOLUTION

By part (1), $\ker\varphi$ is an invariant subspace of the irreducible representation V, so $\ker\varphi = V$ or $\ker\varphi = \mathbf{0}$. In the first case $\varphi = 0$. In the second case φ is injective, so $\mathrm{im}\varphi \cong V$ is nonzero because $V \neq \mathbf{0}$. By part (1), $\mathrm{im}\varphi$ is a nonzero invariant subspace of the irreducible representation W, hence $\mathrm{im}\varphi = W$ and φ is surjective. Thus φ is a bijective G-linear map, which is to say an isomorphism of G-representations. □

3. Suppose V is irreducible and $\varphi : V \to V$ is G-linear. Prove that $\varphi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$, where 1_V is the identity map. Conclude that if V and W are irreducible, then $\dim \mathrm{Hom}_G(V,W)$ is 1 if $V \cong W$ and 0 otherwise. **Hint:** Since $\mathbb{C}$ is algebraically closed, φ has an eigenvalue.

SOLUTION

Since V is a nonzero finite dimensional complex vector space, the characteristic polynomial of φ has a root $\lambda \in \mathbb{C}$, so $\varphi - \lambda 1_V$ has a nonzero kernel. The identity 1_V is G-linear, so $\varphi - \lambda 1_V \in \mathrm{Hom}_G(V,V)$. By Schur's lemma, $\varphi - \lambda 1_V$ is either zero or an isomorphism, and it is not injective, so $\varphi - \lambda 1_V = 0$. Hence $\mathrm{Hom}_G(V,V) = \mathbb{C} \cdot 1_V$ has dimension 1.
Now let V and W be irreducible. If $V \ncong W$, then by Schur's lemma every element of $\mathrm{Hom}_G(V,W)$ is zero, so $\dim \mathrm{Hom}_G(V,W) = 0$. If $\psi : V \to W$ is an isomorphism of G-representations, then ψ^{-1} is G-linear, so for any $\varphi \in \mathrm{Hom}_G(V,W)$ the composite $\psi^{-1}\varphi$ lies

SOLUTION (CONTINUED)

in $\mathrm{Hom}_G(V,V)$ and therefore equals $\lambda 1_V$ for some λ. Then $\varphi = \lambda\psi$, so $\mathrm{Hom}_G(V,W) = \mathbb{C}\psi$, which has dimension 1 since $\psi \neq 0$. □

Schur's lemma is simple to prove but remarkably powerful. Here is a first application.

PROBLEM 6.12

Let G be a finite abelian group, let V be a G-representation, and let $h \in G$. Prove that the map $\rho_h : V \to V$ given by $\rho_h(v) := hv$ is G-linear. Conclude that every irreducible representation of G is 1-dimensional.

SOLUTION

The map ρ_h is linear by the definition of a linear representation. For $g \in G$ and $v \in V$ we have $\rho_h(gv) = hgv = ghv = g\rho_h(v)$, using that G is abelian, so ρ_h is G-linear.
Now suppose V is irreducible. By Problem 6.11(3), $\rho_h = \lambda_h 1_V$ for some scalar $\lambda_h \in \mathbb{C}$, for every $h \in G$. Thus every element of G acts on V by a scalar, so *every* subspace of V is G-invariant. Since $V \neq \mathbf{0}$ we may choose $v \neq 0$ in V, and then $\mathrm{Span}(v)$ is a nonzero invariant subspace, hence $\mathrm{Span}(v) = V$ and $\dim V = 1$. □

6.4 · Uniqueness of Irreducible Decomposition

We can now settle the uniqueness question. The idea is that Schur's lemma lets us *count* how many times an irrep W appears inside V by computing the dimension of $\mathrm{Hom}_G(W,V)$, and this dimension does not depend on any choices.

PROBLEM 6.13

1. Let U, U', W be G-representations. Prove that there is an isomorphism of vector spaces

$$\mathrm{Hom}_G(W, U \oplus U') \cong \mathrm{Hom}_G(W,U) \oplus \mathrm{Hom}_G(W,U').$$

The same argument with the roles reversed shows that $\mathrm{Hom}_G(U \oplus U', W) \cong \mathrm{Hom}_G(U,W) \oplus \mathrm{Hom}_G(U',W)$.

SOLUTION

Let $\pi : U \oplus U' \to U$ and $\pi' : U \oplus U' \to U'$ be the projections $\pi(u \oplus u') = u$ and $\pi'(u \oplus u') = u'$, and let $\iota : U \to U \oplus U'$ and $\iota' : U' \to U \oplus U'$ be the inclusions $\iota(u) = u \oplus 0$ and $\iota'(u') = 0 \oplus u'$. All four maps are G-linear because G acts componentwise on $U \oplus U'$, and they satisfy

SOLUTION (CONTINUED)

$\iota\pi + \iota'\pi' = 1_{U\oplus U'}$.
Define $\Phi : \mathrm{Hom}_G(W, U\oplus U') \to \mathrm{Hom}_G(W,U)\oplus\mathrm{Hom}_G(W,U')$ by $\Phi(\varphi) := \pi\varphi \oplus \pi'\varphi$. Composites of G-linear maps are G-linear, so this makes sense, and Φ is linear since composition is linear in φ. Define Ψ in the other direction by $\Psi(\alpha\oplus\alpha') := \iota\alpha + \iota'\alpha'$, that is, $w \mapsto \alpha(w)\oplus\alpha'(w)$. Then $\Phi\Psi(\alpha\oplus\alpha') = \pi(\iota\alpha+\iota'\alpha')\oplus\pi'(\iota\alpha+\iota'\alpha') = \alpha\oplus\alpha'$ since $\pi\iota = 1_U$, $\pi\iota' = 0$, $\pi'\iota = 0$, and $\pi'\iota' = 1_{U'}$. Also $\Psi\Phi(\varphi) = \iota\pi\varphi + \iota'\pi'\varphi = (\iota\pi+\iota'\pi')\varphi = \varphi$. Hence Φ is an isomorphism.
For the reversed statement, send $\varphi \in \mathrm{Hom}_G(U\oplus U', W)$ to $\varphi\iota\oplus\varphi\iota'$, with inverse $\beta\oplus\beta' \mapsto \beta\pi + \beta'\pi'$; the same identities show these are mutually inverse. □

2. Suppose $V \cong W_1\oplus\cdots\oplus W_\ell$ with each W_i irreducible, and let W be an irreducible representation. Prove that $\dim\mathrm{Hom}_G(W,V)$ and $\dim\mathrm{Hom}_G(V,W)$ both equal the number of indices i with $W_i \cong W$. Conclude that if $V \cong W_1'\oplus\cdots\oplus W_m'$ is another decomposition into irreducibles, then $m = \ell$ and, after reordering, $W_i \cong W_i'$ for all i.

SOLUTION

Let $\psi : V \to W_1\oplus\cdots\oplus W_\ell$ be an isomorphism of G-representations. Composing with ψ gives a linear map $\mathrm{Hom}_G(W,V) \to \mathrm{Hom}_G(W, W_1\oplus\cdots\oplus W_\ell)$, $\varphi \mapsto \psi\varphi$, which is an isomorphism with inverse $\varphi \mapsto \psi^{-1}\varphi$. Applying part (1) repeatedly (by induction on ℓ),

$$\mathrm{Hom}_G(W, W_1\oplus\cdots\oplus W_\ell) \cong \mathrm{Hom}_G(W,W_1)\oplus\cdots\oplus\mathrm{Hom}_G(W,W_\ell),$$

so $\dim\mathrm{Hom}_G(W,V) = \sum_{i=1}^{\ell}\dim\mathrm{Hom}_G(W,W_i)$ by Problem 6.2. By Problem 6.11(3), each term is 1 if $W_i \cong W$ and 0 otherwise, so the sum counts the indices i with $W_i \cong W$. The computation of $\dim\mathrm{Hom}_G(V,W)$ is identical, using the reversed statement in part (1) and composing with ψ^{-1} instead.
For uniqueness, fix an irreducible W. The number of i with $W_i \cong W$ and the number of j with $W_j' \cong W$ are both equal to $\dim\mathrm{Hom}_G(W,V)$, a quantity which depends only on V and W. So the two decompositions contain each isomorphism type of irrep the same number of times. Summing over the isomorphism types which occur gives $\ell = m$, and matching up the W_i and W_j' of each type gives the reordering. □

Given an irreducible representation W, we call

$$m_W(V) := \dim\mathrm{Hom}_G(W,V)$$

the **multiplicity** of W in V. Problem 6.13 says that V is determined up to isomorphism by its multiplicities: if $W_1,\dots,W_r$ are representatives for the isomorphism classes of irreps appearing in V, then

$$V \cong W_1^{\oplus m_1}\oplus\cdots\oplus W_r^{\oplus m_r}, \qquad m_i = m_{W_i}(V),$$

where $W^{\oplus m}$ denotes the direct sum of m copies of W.

The simplest irrep is the trivial representation on $\mathbb{C}$, which we denote by $\mathbf{1}$. Let's compute its multiplicity. If V is a G-representation, the **fixed subspace** of V is

$$V^G := \{v \in V : gv = v \text{ for all } g \in G\}.$$

Since G is finite, we may define the **averaging operator** $P : V \to V$ by

$$Pv := \frac{1}{|G|} \sum_{g \in G} gv.$$

Compare this with the averaged inner product of Problem 6.9: in both cases, averaging over the group produces something G-invariant.

PROBLEM 6.14

Let G be a finite group and let V be a G-representation.

1. Prove that V^G is a G-invariant subspace of V, that P is G-linear, that $\mathrm{im}P \subseteq V^G$, and that $Pv = v$ for every $v \in V^G$. Conclude that $\mathrm{im}P = V^G$ and $P^2 = P$. That is, P is a projection onto V^G.

SOLUTION

Since each g acts linearly, $g0 = 0$ and $g(cv + c'v') = cgv + c'gv' = cv + c'v'$ whenever $v, v' \in V^G$, so V^G is a subspace. It is invariant since $gv = v \in V^G$ for $v \in V^G$.
The map P is linear because each g is. For $h \in G$,

$$hPv = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g' \in G} g'v = Pv,$$

reindexing by $g' = hg$ (left multiplication by h is a bijection $G \to G$), so $Pv \in V^G$, which shows $\mathrm{im}P \subseteq V^G$. Similarly $P(hv) = \frac{1}{|G|} \sum_g ghv = Pv$, reindexing by $g' = gh$. Combining the two, $P(hv) = Pv = hPv$, so P is G-linear. If $v \in V^G$, then every term gv equals v, so $Pv = \frac{1}{|G|} \cdot |G|v = v$. Hence $V^G \subseteq \mathrm{im}P$, so $\mathrm{im}P = V^G$, and $P(Pv) = Pv$ because $Pv \in V^G$, so $P^2 = P$. □

2. Prove that the map $\mathrm{Hom}_G(\mathbf{1}, V) \to V^G$ given by $\varphi \mapsto \varphi(1)$ is an isomorphism of vector spaces. Conclude that $\dim V^G = m_{\mathbf{1}}(V)$ is the multiplicity of the trivial representation in V.

SOLUTION

If $\varphi \in \mathrm{Hom}_G(\mathbf{1}, V)$ then $g\varphi(1) = \varphi(g \cdot 1) = \varphi(1)$ since G acts trivially on $\mathbf{1}$, so $\varphi(1) \in V^G$ and the map is well defined. It is linear since $(c\varphi + c'\psi)(1) = c\varphi(1) + c'\psi(1)$. It is injective because a linear map $\varphi : \mathbb{C} \to V$ is determined by $\varphi(1)$, namely $\varphi(z) = z\varphi(1)$. It is surjective: given $v \in V^G$, define $\varphi(z) := zv$. This is linear, and it is G-linear since $\varphi(gz) =$

SOLUTION (CONTINUED)

$\varphi(z) = zv$ while $g\varphi(z) = zgv = zv$. Thus $\mathrm{Hom}_G(\mathbf{1}, V) \cong V^G$, and taking dimensions gives $m_{\mathbf{1}}(V) = \dim V^G$. □

3. Let X be a finite G-set. For each orbit $O \subseteq X$ let $e_O := \sum_{x \in O} e_x \in \mathbb{C}X$. Prove that the vectors e_O, one for each orbit, form a basis for $(\mathbb{C}X)^G$. Hence $\dim(\mathbb{C}X)^G$ is the number of orbits of G on X.

SOLUTION

For $g \in G$ we have $ge_O = \sum_{x \in O} e_{gx} = e_O$, since $x \mapsto gx$ is a bijection from O to itself. So each e_O lies in $(\mathbb{C}X)^G$.
The orbits are pairwise disjoint, so distinct e_O involve disjoint sets of basis vectors e_x. If $\sum_O c_O e_O = 0$, then looking at the coefficient of e_x for any $x \in O$ gives $c_O = 0$, so the e_O are linearly independent.
For spanning, let $v = \sum_{x \in X} c_x e_x \in (\mathbb{C}X)^G$. For $g \in G$,

$$gv = \sum_{x \in X} c_x e_{gx} = \sum_{x \in X} c_{g^{-1}x} e_x,$$

reindexing by $x \mapsto g^{-1}x$. Comparing coefficients in $gv = v$ gives $c_{g^{-1}x} = c_x$ for all $g \in G$ and $x \in X$, so the coefficients c_x are constant on each orbit. Writing c_O for the common value on O, we get $v = \sum_O c_O e_O$. □

We can squeeze a little more out of the averaging operator by computing its trace. Recall that the **trace** of a square matrix $A = (a_{ij})$ is the sum of its diagonal entries, $\mathrm{tr}(A) := \sum_i a_{ii}$, and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any square matrices A, B of the same size. Consequently $\mathrm{tr}(Q^{-1}AQ) = \mathrm{tr}(AQQ^{-1}) = \mathrm{tr}(A)$, so the trace of a linear map $T : V \to V$ may be defined as the trace of its matrix with respect to any basis. Note also that tr is linear: $\mathrm{tr}(cA + c'B) = c\,\mathrm{tr}(A) + c'\,\mathrm{tr}(B)$.

PROBLEM 6.15

1. Let $P : V \to V$ be a linear map with $P^2 = P$. Prove that $V = \mathrm{im}P \oplus \ker P$, in the sense that $\mathrm{im}P$ and $\ker P$ satisfy the internal direct sum criteria, and deduce that $\mathrm{tr}(P) = \dim \mathrm{im}P$. **Hint:** $v = Pv + (v - Pv)$.

SOLUTION

For any $v \in V$ we have $v = Pv + (v - Pv)$ with $Pv \in \mathrm{im}P$ and $P(v - Pv) = Pv - P^2v = 0$, so $v - Pv \in \ker P$. Hence $\mathrm{im}P + \ker P = V$. If $v \in \mathrm{im}P \cap \ker P$, write $v = Pu$; then $v = Pu = P^2u = Pv = 0$. So $\mathrm{im}P \cap \ker P = \mathbf{0}$ and the internal direct sum criteria hold.
By Problems 6.2 and 6.3, if $v_1, \ldots, v_r$ is a basis of $\mathrm{im}P$ and $v_{r+1}, \ldots, v_n$ is a basis of $\ker P$,

SOLUTION (CONTINUED)

then $v_1, \dots, v_n$ is a basis of V. For $i \le r$ write $v_i = Pu_i$; then $Pv_i = P^2u_i = Pu_i = v_i$, while $Pv_i = 0$ for $i > r$. So the matrix of P in this basis is diagonal with r ones followed by $n - r$ zeros, and $\operatorname{tr}(P) = r = \dim \operatorname{im} P$. □

2. Let X be a finite G-set and let $g \in G$. Write $X^g := \{x \in X : gx = x\}$ for the set of fixed points of g. Prove that the trace of g acting on $\mathbb{C}X$ equals $|X^g|$. Then prove **Burnside's lemma**: the number of orbits of G on X is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|.$$

SOLUTION

Compute the matrix of g with respect to the basis $\{e_x : x \in X\}$ of $\mathbb{C}X$. The column indexed by x records $ge_x = e_{gx}$, so it has a single 1 in the row indexed by gx and zeros elsewhere. The diagonal entry in row and column x is therefore 1 if $gx = x$ and 0 otherwise. Summing the diagonal entries gives $\operatorname{tr}(g) = |X^g|$.
Now let P be the averaging operator on $\mathbb{C}X$. By linearity of the trace,

$$\operatorname{tr}(P) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g) = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

On the other hand $P^2 = P$ and $\operatorname{im} P = (\mathbb{C}X)^G$ by Problem 6.14(1), so by part (1) and Problem 6.14(3),

$$\operatorname{tr}(P) = \dim(\mathbb{C}X)^G = \text{the number of orbits of } G \text{ on } X.$$

Equating the two expressions for $\operatorname{tr}(P)$ proves Burnside's lemma. □

The function $g \mapsto \operatorname{tr}(g)$ sending each group element to the trace of its action on V is called the **character** of V. Burnside's lemma is our first glimpse of how much information the character carries; we will study characters more in depth in the coming weeks.

6.6 · The Regular Representation

Recall from the last problem set that the **regular representation** $\mathbb{C}G$ has basis $\{e_h : h \in G\}$ with $ge_h = e_{gh}$. We claimed then that $\mathbb{C}G$ contains a copy of every representation we could ever want. Let's make that precise. Throughout, ε denotes the identity element of G.

PROBLEM 6.16

Let G be a finite group.

1. Let W be any G-representation. Prove that the map $\mathrm{Hom}_G(\mathbb{C}G, W) \to W$ given by $\varphi \mapsto \varphi(e_\varepsilon)$ is an isomorphism of vector spaces.

SOLUTION

The map is linear since $(c\varphi + c'\psi)(e_\varepsilon) = c\varphi(e_\varepsilon) + c'\psi(e_\varepsilon)$. It is injective: if φ is G-linear, then $\varphi(e_h) = \varphi(he_\varepsilon) = h\varphi(e_\varepsilon)$ for every $h \in G$, so φ is determined on the basis $\{e_h\}$, and hence everywhere, by the single vector $\varphi(e_\varepsilon)$. It is surjective: given $w \in W$, define $\varphi : \mathbb{C}G \to W$ on the basis by $\varphi(e_h) := hw$ and extend linearly, so $\varphi(\sum_h c_h e_h) = \sum_h c_h hw$. Then φ is G-linear, since it suffices to check on basis vectors and $\varphi(ge_h) = \varphi(e_{gh}) = ghw = g\varphi(e_h)$. Finally $\varphi(e_\varepsilon) = \varepsilon w = w$. □

2. Prove that every irreducible representation W of G occurs in $\mathbb{C}G$ with multiplicity $m_W(\mathbb{C}G) = \dim W$. Use this to show that G has only finitely many irreducible representations up to isomorphism, and that

$$\sum_W (\dim W)^2 = |G|,$$

where the sum runs over the irreducible representations of G up to isomorphism.

SOLUTION

By Problem 6.13(2), the multiplicity of W in $\mathbb{C}G$ may be computed as $\dim \mathrm{Hom}_G(\mathbb{C}G, W)$, which equals $\dim W$ by part (1). In particular every irrep W has $m_W(\mathbb{C}G) = \dim W \geq 1$, so every irrep appears in a decomposition $\mathbb{C}G \cong W_1 \oplus \cdots \oplus W_\ell$ into irreducibles. Thus every irrep of G is isomorphic to one of the finitely many W_i, and there are only finitely many irreps up to isomorphism. Grouping the W_i by isomorphism type and taking dimensions,

$$|G| = \dim \mathbb{C}G = \sum_{i=1}^{\ell} \dim W_i = \sum_W m_W(\mathbb{C}G) \dim W = \sum_W (\dim W)^2. \qquad \square$$

3. Let $G = S_3$. Let $\mathbf{1}$ be the trivial representation, let $\mathbb{C}_{\mathrm{sgn}}$ be the **sign representation** on $\mathbb{C}$ given by $\sigma z := \mathrm{sgn}(\sigma) z$, and let V_2 be the plane from the beginning of this problem set, which we showed is irreducible. Prove that $\mathbf{1}$, $\mathbb{C}_{\mathrm{sgn}}$, and V_2 are pairwise non-isomorphic and that every irreducible representation of S_3 is isomorphic to one of them. Conclude that

$$\mathbb{C}S_3 \cong \mathbf{1} \oplus \mathbb{C}_{\mathrm{sgn}} \oplus V_2 \oplus V_2,$$

and find explicit vectors $u, u' \in \mathbb{C}S_3$ such that $\mathrm{Span}(u) \cong \mathbf{1}$ and $\mathrm{Span}(u') \cong \mathbb{C}_{\mathrm{sgn}}$.

SOLUTION

The representations $\mathbf{1}$ and $\mathbb{C}_{\mathrm{sgn}}$ are 1-dimensional and given by different homomorphisms $S_3 \to \mathbb{C}^\times$ (they differ on (12)), so they are not isomorphic: a G-linear isomorphism $\varphi : \mathbf{1} \to \mathbb{C}_{\mathrm{sgn}}$ is multiplication by some $c \neq 0$, and then $c = \varphi((12) \cdot 1) = (12) \cdot \varphi(1) = -c$,

SOLUTION (CONTINUED)

a contradiction. Neither is isomorphic to V_2, which has dimension 2. All three are irreducible: 1-dimensional representations trivially so, and V_2 by the discussion following the definition of irreducibility. Now $(\dim \mathbf{1})^2+(\dim \mathbb{C}_{\mathrm{sgn}})^2+(\dim V_2)^2 = 1+1+4 = 6 = |S_3|$, so by part (2) any further irrep would make the sum of squares exceed $|S_3|$. Hence these are all the irreps of S_3. By part (2) their multiplicities in $\mathbb{C}S_3$ are 1, 1, and 2, which gives the stated decomposition.

Let $u := \sum_{\sigma\in S_3} e_\sigma$ and $u' := \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)e_\sigma$. For $\tau \in S_3$, reindexing by $\sigma' = \tau\sigma$,

$$\tau u = \sum_{\sigma} e_{\tau\sigma} = u \qquad \text{and} \qquad \tau u' = \sum_{\sigma} \mathrm{sgn}(\sigma)e_{\tau\sigma} = \sum_{\sigma'} \mathrm{sgn}(\tau^{-1}\sigma')e_{\sigma'} = \mathrm{sgn}(\tau)u',$$

using $\mathrm{sgn}(\tau^{-1}\sigma') = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma')$. So $\mathrm{Span}(u)$ is a copy of $\mathbf{1}$ and $\mathrm{Span}(u')$ is a copy of $\mathbb{C}_{\mathrm{sgn}}$. Note that u spans $(\mathbb{C}S_3)^{S_3}$ by Problem 6.14(3), since S_3 acting on itself has a single orbit. With respect to the standard inner product on $\mathbb{C}S_3$ (which is S_3-invariant, since each τ permutes the orthonormal basis $\{e_\sigma\}$), the orthogonal complement of $\mathrm{Span}(u, u')$ is a 4-dimensional subrepresentation isomorphic to $V_2 \oplus V_2$. □

The identity $\sum_W (\dim W)^2 = |G|$ is a powerful constraint. For instance, combined with Problem 6.12 it shows that a finite abelian group G has exactly $|G|$ irreducible representations, and conversely that a group with an irreducible representation of dimension greater than 1 cannot be abelian.

6.7 · Challenge: Doubly Transitive Actions

The following is a challenge problem for those who want to push these ideas further. We will use the counting tools developed above to build new irreducible representations out of permutation representations. If X and Y are G-sets, then G acts on $Y \times X$ by $g(y, x) := (gy, gx)$.

PROBLEM 6.17 · (Challenge)

1. Let X and Y be finite G-sets. A linear map $\varphi : \mathbb{C}X \to \mathbb{C}Y$ is determined by the numbers $a_{y,x} \in \mathbb{C}$ defined by $\varphi(e_x) = \sum_{y\in Y} a_{y,x}e_y$. Prove that φ is G-linear if and only if $a_{gy,gx} = a_{y,x}$ for all $g \in G$, $x \in X$, $y \in Y$. Conclude that $\dim \mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}Y)$ is the number of orbits of G on $Y \times X$.

SOLUTION

For $g \in G$ and $x \in X$ we have $\varphi(ge_x) = \varphi(e_{gx}) = \sum_y a_{y,gx}e_y$, while

$$g\varphi(e_x) = \sum_y a_{y,x}e_{gy} = \sum_y a_{g^{-1}y,x}e_y.$$

So φ is G-linear if and only if $a_{y,gx} = a_{g^{-1}y,x}$ for all g, x, y; replacing y by gy, this says

SOLUTION (CONTINUED)

$a_{gy,gx} = a_{y,x}$. Thus $\varphi \mapsto (a_{y,x})$ identifies $\mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}Y)$ with the space of functions $Y \times X \to \mathbb{C}$ that are constant on G-orbits, which has a basis given by the indicator functions of the orbits. (Equivalently, $\varphi \mapsto \sum a_{y,x} e_{(y,x)}$ identifies $\mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}Y)$ with $(\mathbb{C}(Y \times X))^G$, and we may quote Problem 6.14(3).) □

2. Let V and V' be G-representations. Prove that

$$\dim \mathrm{Hom}_G(V, V') = \sum_W m_W(V)\, m_W(V'),$$

where the sum runs over the irreps of G up to isomorphism. Deduce that V is irreducible if and only if $\dim \mathrm{Hom}_G(V, V) = 1$.

SOLUTION

Decompose $V \cong \bigoplus_i W_i$ and $V' \cong \bigoplus_j W'_j$ into irreducibles. Applying both halves of Problem 6.13(1) repeatedly,

$$\mathrm{Hom}_G(V, V') \cong \bigoplus_{i,j} \mathrm{Hom}_G(W_i, W'_j),$$

and by Problem 6.11(3) the (i, j) summand has dimension 1 if $W_i \cong W'_j$ and 0 otherwise. For each irrep W there are $m_W(V)\, m_W(V')$ pairs (i, j) with $W_i \cong W \cong W'_j$, giving the formula. Taking $V' = V$ gives $\dim \mathrm{Hom}_G(V, V) = \sum_W m_W(V)^2$, which equals 1 exactly when a single irrep occurs, with multiplicity 1. □

3. Let X be a G-set with $|X| \geq 2$. We say G acts **doubly transitively** on X if for any $x \neq y$ and $x' \neq y'$ in X there is some $g \in G$ with $gx = x'$ and $gy = y'$. Prove that the following are equivalent:

   (a) G acts doubly transitively on X;

   (b) G has exactly two orbits on $X \times X$;

   (c) $\mathbb{C}X \cong \mathbf{1} \oplus W$ where W is irreducible and $W \not\cong \mathbf{1}$.

   Show that in this case W may be taken to be $\{\sum_x c_x e_x : \sum_x c_x = 0\}$.

SOLUTION

The diagonal $\Delta = \{(x, x)\}$ and its complement are both unions of orbits, and both are nonempty since $|X| \geq 2$. Double transitivity says precisely that $X \times X \setminus \Delta$ is a single orbit; it also forces G to be transitive (given $x \neq x'$, choose $y \neq x$ and $y' \neq x'$, and map $(x, y) \mapsto (x', y')$), so Δ is a single orbit as well. Conversely two orbits means Δ and its complement are each an orbit. So (a) $\iff$ (b).
By parts (1) and (2), the number of orbits on $X \times X$ is $\sum_W m_W(\mathbb{C}X)^2$. If this is 2, then exactly

SOLUTION (CONTINUED)

two irreps occur, each with multiplicity 1. One of them is $\mathbf{1}$, since $m_{\mathbf{1}}(\mathbb{C}X) = \dim(\mathbb{C}X)^G$ is the number of orbits on X, which is at least 1; this gives (c). Conversely, (c) gives $\sum_W m_W(\mathbb{C}X)^2 = 1 + 1 = 2$. So (b) $\iff$ (c).
The standard inner product on $\mathbb{C}X$ is G-invariant since G permutes the orthonormal basis $\{e_x\}$. Since G is transitive, $(\mathbb{C}X)^G = \mathrm{Span}(e_X)$ with $e_X = \sum_x e_x$, and its orthogonal complement is the sum-zero subspace, which is a G-invariant complement to $\mathbf{1}$, hence isomorphic to W by Problem 6.13. □

4. Applications.

   (a) Prove that for $n \geq 2$ the **standard representation** $V_{n-1} := \{(c_1, \ldots, c_n) \in \mathbb{C}^n : \sum_i c_i = 0\}$ of S_n is irreducible. (For $n = 3$ this is the plane V_2 from the beginning of this problem set.)

   (b) Prove that for $n \geq 4$, V_{n-1} remains irreducible when regarded as a representation of the alternating group A_n. What goes wrong when $n = 3$?

   (c) Let p be a prime and let $G = \mathrm{Aff}(\mathbb{F}_p)$ be the group of functions $\mathbb{F}_p \to \mathbb{F}_p$ of the form $x \mapsto ax + b$ with $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$, under composition. Determine all irreducible representations of G up to isomorphism: prove that there are exactly p of them, namely $p - 1$ of dimension 1 and one of dimension $p - 1$. **Hint:** Look at the map $G \to \mathbb{F}_p^\times$ sending $x \mapsto ax + b$ to a, and recall the remark following Problem 6.16. (When $p = 3$, check that $G \cong S_3$ and you recover Problem 6.16(3).)

   (d) Let $n \geq 4$ and let X be the set of 2-element subsets of $[n]$, with the natural action of S_n. Prove that $\mathbb{C}X \cong \mathbf{1} \oplus V_{n-1} \oplus U$ where U is an irreducible representation of S_n of dimension $n(n-3)/2$, not isomorphic to $\mathbf{1}$ or V_{n-1}. In particular, this produces a 2-dimensional irrep of S_4 and a 5-dimensional irrep of S_5. **Hint:** Use part (1) to compute $\dim \mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}X)$ and $\dim \mathrm{Hom}_G(\mathbb{C}[n], \mathbb{C}X)$.

SOLUTION

(a) S_n acts doubly transitively on $[n]$: given $i \neq j$ and $k \neq \ell$, extend $i \mapsto k$, $j \mapsto \ell$ by any bijection $[n] \setminus \{i, j\} \to [n] \setminus \{k, \ell\}$ to get a permutation. Apply part (3).
(b) Given $i \neq j$ and $k \neq \ell$, choose $\sigma \in S_n$ as in (a). If σ is odd, since $n \geq 4$ there are two elements $r, s \notin \{k, \ell\}$, and $(r\,s)\sigma$ is even and still sends $i \mapsto k$, $j \mapsto \ell$. So A_n is doubly transitive and part (3) applies. For $n = 3$, $A_3 \cong C_3$ is abelian, so by Problem 6.12 its irreps are 1-dimensional and V_2 must split; indeed A_3 has three orbits on $[3] \times [3]$ (the diagonal, and the two cyclic orientations $\{(1,2),(2,3),(3,1)\}$ and $\{(2,1),(3,2),(1,3)\}$).
(c) $|G| = p(p-1)$. G acts doubly transitively on $\mathbb{F}_p$: given $x \neq y$ and $x' \neq y'$, the equations $ax + b = x'$, $ay + b = y'$ have the solution $a = (x' - y')/(x - y) \neq 0$, $b = x' - ax$. By part (3), $\mathbb{C}\mathbb{F}_p \cong \mathbf{1} \oplus W$ with W irreducible of dimension $p - 1$. Next, $(x \mapsto ax + b) \circ (x \mapsto a'x + b') = (x \mapsto aa'x + (ab' + b))$, so $\pi : G \to \mathbb{F}_p^\times$, $(x \mapsto ax + b) \mapsto a$, is a surjective homomorphism. Since $\mathbb{F}_p^\times$ is abelian of order $p - 1$, it has $p - 1$ pairwise non-isomorphic irreps, each a homomorphism $\chi : \mathbb{F}_p^\times \to \mathbb{C}^\times$. The composites $\chi \circ \pi$ are 1-dimensional

SOLUTION (CONTINUED)

representations of G, and they are pairwise non-isomorphic: an isomorphism between 1-dimensional representations is multiplication by some $c \neq 0$, which forces $\chi \circ \pi = \chi' \circ \pi$ as functions, hence $\chi = \chi'$ since π is surjective. Now $(p-1)\cdot 1^2 + (p-1)^2 = p(p-1) = |G|$, so by Problem 6.16(2) these are all the irreps of G. (When $p > 2$, W is not among the 1-dimensional ones by dimension; when $p = 2$, $G \cong C_2$ and W is the sign representation, which is the one 1-dimensional representation not of the form $\chi \circ \pi$.)
(d) Orbits of S_n on $X \times X$ are classified by $|A \cap B| \in \{2, 1, 0\}$, and all three occur since $n \geq 4$; so $\sum_W m_W(\mathbb{C}X)^2 = 3$, and $\mathbb{C}X$ is a sum of three pairwise non-isomorphic irreps, each with multiplicity 1. One is $\mathbf{1}$ since S_n is transitive on X. Orbits of S_n on $X \times [n]$ are classified by whether $i \in A$, so by parts (1) and (2), using $\mathbb{C}[n] \cong \mathbf{1} \oplus V_{n-1}$ from (a),

$$2 = \dim \mathrm{Hom}_G(\mathbb{C}[n], \mathbb{C}X) = m_{\mathbf{1}}(\mathbb{C}X) + m_{V_{n-1}}(\mathbb{C}X) = 1 + m_{V_{n-1}}(\mathbb{C}X).$$

So V_{n-1} is the second irrep, and the third U has dimension $\binom{n}{2} - 1 - (n-1) = n(n-3)/2$. □