MATH 301 · FALL 2026

Problem Set 7: Characters

DUE Friday, November 13th by 11:59pm

Laughter and tears are both responses to frustration and exhaustion. I myself prefer to laugh, since there is less cleaning up to do afterward.

Kurt Vonnegut

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

7.1 · Trace

Let's recall the definition of the *trace* of a matrix.

DEFINITION 7.1
Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. The **trace** of A is $\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}$, the sum of the diagonal entries of A.

PROBLEM 7.1

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices.

1. Prove that $\operatorname{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is $\mathbb{C}$-linear.
2. Let $A, B \in M_n(\mathbb{C})$. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Use this to prove that $\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B)$. Hence trace is conjugation invariant.
3. Suppose A is diagonalizable, meaning that there is some invertible matrix B such that BAB^{-1} is a diagonal matrix. Prove that $\operatorname{tr}(A)$ is the sum of the eigenvalues of A (with multiplicity).

7.2 · Characters

Now suppose that $\rho_V : G \to \operatorname{Aut}(V)$ is a finite dimensional representation of a finite group G.

DEFINITION 7.2
The **character** of V is the function $\chi_V : G \to \mathbb{C}$ defined by

$$\chi_V(g) := \operatorname{tr} \rho_V(g).$$

A representation of G is a collection of matrices, one for each element of the group. That's a lot of data. The character χ_V assigns a single number to each element of the group. Hence characters are a method of compressing a representation into a simple list of numbers. What's incredible is that the character of a representation V contains enough information to determine V up to isomorphism. This will take us some work to prove.

PROBLEM 7.2

Let $\rho_V : G \to \mathrm{Aut}(V)$ be a finite dimensional representation of a finite group G.

1. Prove that $\chi_V(1) = \dim V$.
2. Suppose that U is another finite dimensional representation of G. Prove that $\chi_{U\oplus V} = \chi_U + \chi_V$.
3. Prove that $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, where $\overline{a+bi} = a - bi$ is the complex conjugate function.
4. Suppose $V \cong W$ are isomorphic G-linear representations. Prove that $\chi_V = \chi_W$.

On the last problem set we proved that every finite dimensional G-representation V decomposes as a direct sum of irreps. Thus Problem 7.2.2 implies that every character of G may be expressed as an $\mathbb{N}$-linear combination of irreducible characters (i.e. characters of irreps.)

A **class function** is any function $f : G \to \mathbb{C}$ which satisfies $f(ghg^{-1}) = f(h)$. In other words, a class function is any function on G which is constant on conjugacy classes. Let $\mathrm{CF}(G)$ be the set of all complex-valued class functions defined on G. Note that the conjugation invariance of trace proved in Problem 7.1.2 implies that $\chi_V(ghg^{-1}) = \chi_V(h)$, hence that $\chi_V \in \mathrm{CF}(G)$. With respect to pointwise operations, $\mathrm{CF}(G)$ forms a vector space with dimension equal to the number of conjugacy classes of elements in G.

PROBLEM 7.3

Let $f_1, f_2 \in \mathrm{CF}(G)$ be class functions. Define

$$\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Prove that $\langle \cdot\,, \cdot\rangle$ is an inner product on $\mathrm{CF}(G)$.

7.3 · Row Orthogonality

Our goal in this section is to prove that the characters of irreps of G are orthogonal to one another with respect to the inner product of class functions defined above.

Suppose now that V, W are finite dimensional G-representations. Let $\mathrm{Hom}(V, W)$ be the set of $\mathbb{C}$-linear maps from V to W (not just the G-linear maps!). Note that $\mathrm{Hom}(V, W)$ is a $\mathbb{C}$-vector space with operations defined pointwise. The action of G on V and W induces a linear action on $\mathrm{Hom}(V, W)$ defined by

$$g \cdot T := \rho_W(g) \circ T \circ \rho_V(g^{-1}).$$

PROBLEM 7.4

Let V and W be finite dimensional representations of a finite group G.

1. Recall that $V^G \subseteq V$ is the subspace of G-invariant vectors in V. Prove that
$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

2. Prove that
$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_W(g)\overline{\chi_V(g)}.$$
This one requires careful and disciplined maneuvering with indices.

3. Prove that $\langle \chi_W, \chi_V \rangle = \dim \mathrm{Hom}_G(V, W)$.

4. Suppose that V, W are irreps of G. Prove that
$$\langle \chi_W, \chi_V \rangle = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise.} \end{cases}$$

5. Suppose that V is a finite dimensional representation of G. Prove that V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Given a finite group G, we proved on the last problem set that up to isomorphism there are finitely many irreducible representations of G, hence finitely many irreducible characters. We traditionally organize these irreducible characters into a **character table**. The rows of the character table correspond to the irreps of G and the columns correspond to the conjugacy classes of elements of G. In the entry corresponding to the character χ_V and the conjugacy class of an element g we record the value $\chi_V(g)$.

PROBLEM 7.5

Consider $G = S_3$. We previously proved that G has three irreps, the trivial representation **1**, the standard representation **Std**, and the sign representation **Sgn**. (On the last problem set we called these V_2 and $\mathbb{C}_{\mathrm{sgn}}$; recall that **Std** $= V_2$ is the plane of vectors in $\mathbb{C}^3$ whose coordinates sum to zero.) There are three conjugacy classes of elements in S_3 represented by the identity ε, 2-cycles (12), and 3-cycles (123). Fill in the entries of the character table.

S_3	ε	(12)	(123)
1			
Std			
Sgn			

Problem 7.4 implies that the rows of the character table are orthogonal with respect to the inner product we constructed on $\mathrm{CF}(G)$. This doesn't quite look right when we inspect the character

table of S_3. For example, the rows corresponding to the trivial and sign representations do not look orthogonal. But keep in mind that the entries of this table only record the value of each character on conjugacy classes, while the inner product involves a sum over all elements of the group. Hence when calculating the inner product, we need to weight entries by the size of the conjugacy class. Thus, for example,

$$\langle \chi_{\mathbf{1}}, \chi_{\mathbf{Sgn}} \rangle = \frac{1}{6}(1 \cdot \chi_{\mathbf{1}}(\varepsilon)\overline{\chi_{\mathbf{Sgn}}(\varepsilon)} + 3 \cdot \chi_{\mathbf{1}}(12)\overline{\chi_{\mathbf{Sgn}}(12)} + 2 \cdot \chi_{\mathbf{1}}(123)\overline{\chi_{\mathbf{Sgn}}(123)}) = \frac{1}{6}(1 - 3 + 2) = 0.$$

7.4 · Column Orthogonality

There is also a sense in which the columns of the character table of a finite group G are orthogonal. Note that the columns correspond to conjugacy classes of elements; the entries in the column tell us the values of the irreducible characters on the given conjugacy class.

Consider the function $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{C}$ defined by

$$\langle g, h \rangle = \#\{k \in G : kgk^{-1} = h\}.$$

We notate this suggestively to look like an inner product, though G is not a vector space, so it doesn't quite make sense to talk about inner products on G. The function $\langle g, h \rangle$ essentially detects whether or not g and h are conjugate.

PROBLEM 7.6

Let G be a finite group, let $\rho_i : G \to \mathrm{Aut}(W_i)$ for $1 \le i \le n$ be a complete list of the irreps of G up to isomorphism, let χ_i be the irreducible character of W_i, and let $d_i = \chi_i(1)$ be the dimension of W_i.

1. For $g \in G$, let $C_G(g) := \{k \in G : kgk^{-1} = g\}$ be the **centralizer** of g. Prove that for all $g, h \in G$,

$$\langle g, h \rangle = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate in } G, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let ρ_{reg} be the regular representation of G with character χ_{reg}. See Problem Set 5 for the definition of regular representation. Prove that

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $g, h \in G$. Prove that

$$\langle g, h \rangle = \frac{1}{|G|} \sum_{i=1}^{n} d_i \sum_{k \in G} \chi_i(kgk^{-1}h^{-1}).$$

4. Given $g \in G$, consider the linear transformation $A_{i,g} : W_i \to W_i$ defined by

$$A_{i,g} := \frac{1}{|G|} \sum_{k \in G} \rho_i(kgk^{-1}).$$

Prove that $A_{i,g} = \frac{\chi_i(g)}{d_i} I_{d_i}$, where I_{d_i} is the identity matrix on W_i. **Hint: Schur's lemma.**

5. Prove that

$$\langle g, h \rangle = \sum_{i=1}^{n} \chi_i(g)\overline{\chi_i(h)}.$$

6. Let $K \subseteq G$ be a conjugacy class and let $\delta_K : G \to \mathbb{C}$ be the **indicator function** of K, defined by $\delta_K(h) := 1$ if $h \in K$ and $\delta_K(h) := 0$ otherwise. Prove that for any $g \in K$,

$$\delta_K = \frac{|K|}{|G|} \sum_{i=1}^{n} \overline{\chi_i(g)}\chi_i.$$

Conclude that the irreducible characters $\chi_1, \ldots, \chi_n$ form a basis for $\mathrm{CF}(G)$, and hence that the number of irreducible characters of G is equal to the number of conjugacy classes of elements in G.

The columns of the character table are parametrized by (conjugacy classes of) elements of G. Observe that column g is precisely $(\chi_1(g), \chi_2(g), \ldots, \chi_n(g))$. Hence Problem 7.6.5 is precisely saying that the columns of the character table of G are orthogonal with respect to the standard inner product on $\mathbb{C}^n$. Problem 7.6.6 tells us that the character table always has the same number of rows as columns: the character table of a finite group is square.

7.5 · Characters Determine Representations

We can now make good on the promise from the beginning of this problem set: the character of a representation determines the representation up to isomorphism. Recall from the last problem set that if W is an irrep of G and V is any finite dimensional G-representation, then the **multiplicity** of W in V is $m_W(V) := \dim \mathrm{Hom}_G(W, V)$, and that V is determined up to isomorphism by its multiplicities.

PROBLEM 7.7

Let V be a finite dimensional representation of a finite group G.

1. Let W be an irrep of G. Prove that $m_W(V) = \langle \chi_V, \chi_W \rangle$. Thus the multiplicity of W in V can be computed directly from characters.

2. Let V' be another finite dimensional representation of G. Prove that $V \cong V'$ if and only if $\chi_V = \chi_{V'}$.

Problem 7.7 is the payoff of this problem set. A finite dimensional representation of G is a large amount of data, but its character is just one number for each conjugacy class, and those numbers

already determine the representation up to isomorphism. Better still, the multiplicity formula in Problem 7.7.1 means we can decompose a representation into irreps by computing a few inner products. On the next problem set we will put this to work.