MATH 301 · FALL 2026

Problem Set 8: Character Tables

DUE Sunday, November 22nd by 11:59pm

The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better,and otherwise you let time pass. The shell becomes more flexible through weeks and months – when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!

From *Récoltes et semailles* by Alexander Grothendieck

Solutions must be typeset in LaTeX and written in complete, grammatical sentences. **You must list all your collaborators explicitly**. Feel free to DM on Discord if you have any questions. Problems marked (Challenge) are optional but encouraged.

On the last problem set we proved that the character of a representation determines it up to isomorphism, and that the multiplicity of an irrep W in a representation V is $m_W(V) = \langle \chi_V, \chi_W \rangle$. On this problem set we put these results to work. Our goal is to compute character tables: to find all of the irreducible characters of a group G, usually without ever writing down the irreps themselves. Along the way we will develop a few tools for producing new representations, culminating in the tensor product.

8.1 · Permutation Characters

Let X be a finite G-set. In Problem 6.15.2 we proved that the trace of g acting on the permutation representation $\mathbb{C}X$ is the number of fixed points of g. In the language of characters,

$$\chi_{\mathbb{C}X}(g) = |X^g|, \qquad \text{where } X^g := \{x \in X : gx = x\}.$$

We call $\chi_{\mathbb{C}X}$ the **permutation character** of X.

Recall from Problem 4.4 that up to isomorphism there are four transitive S_3-sets, which we called

$$1 = S_3/S_3, \quad T_2 = S_3/\langle(123)\rangle, \quad T_3 = S_3/\langle(12)\rangle, \quad T_6 = S_3/\langle\varepsilon\rangle.$$

Recall also the character table of S_3 which you computed in Problem 7.5.

S_3	ε	(12)	(123)
1	1	1	1
Std	2	0	-1
Sgn	1	-1	1

PROBLEM 8.1

1. Compute the permutation characters of the four transitive S_3-sets and record them in the table below.

S_3	ε	(12)	(123)
$\mathbb{C}1$			
$\mathbb{C}T_2$			
$\mathbb{C}T_3$			
$\mathbb{C}T_6$			

2. Express each of $\mathbb{C}1$, $\mathbb{C}T_2$, $\mathbb{C}T_3$, and $\mathbb{C}T_6$ as a direct sum of irreducible representations. **Hint:** Characters are all you need. Compare your answer for $\mathbb{C}T_6$ with Problem 6.16.3.

3. Let $X := T_6 + 2$ and $Y := T_2 + 2T_3$. Prove that X and Y are not isomorphic as S_3-sets, but that $\mathbb{C}X$ and $\mathbb{C}Y$ are isomorphic as S_3-representations. Thus passing from a G-set to its permutation representation loses information. Food for thought: what is the information that is lost?

Now let's see what characters tell us about permutation representations in general. If X and Y are G-sets, recall that G acts on $X \times Y$ by $g(x, y) := (gx, gy)$.

PROBLEM 8.2

Let X and Y be finite G-sets.

1. Prove that $\chi_{\mathbb{C}(X\times Y)} = \chi_{\mathbb{C}X}\chi_{\mathbb{C}Y}$.

2. Prove that $\langle \chi_{\mathbb{C}X}, \chi_{\mathbb{C}Y}\rangle$ is the number of orbits of G on $X \times Y$. Conclude that $\dim \mathrm{Hom}_G(\mathbb{C}X, \mathbb{C}Y)$ is the number of orbits of G on $X \times Y$. (Compare this with the challenge Problem 6.17.1.)

3. Suppose X is transitive with $|X| \geq 2$. Let $u := \sum_{x\in X} e_x$ and let $V := \{\sum_{x\in X} a_x e_x : \sum_{x\in X} a_x = 0\}$. Prove that $\mathrm{Span}(u)$ and V are subrepresentations of $\mathbb{C}X$ and that $\mathbb{C}X = \mathrm{Span}(u) \oplus V$.

4. We say G acts **doubly transitively** on X if for all pairs $(x, y), (x', y') \in X \times X$ with $x \neq y$ and $x' \neq y'$ there is some $g \in G$ with $gx = x'$ and $gy = y'$. Continue with the assumptions of the previous part. Prove that V is irreducible if and only if G acts doubly transitively on X.

5. Prove that for all $n \geq 2$, the **standard representation**

$$\mathbf{Std}_n := \{(a_1, \ldots, a_n) \in \mathbb{C}^n : a_1 + \cdots + a_n = 0\}$$

of S_n is irreducible.

8.2 · Commutators

The 1-dimensional representations of a group are the easiest to find, and they will be the first rows we fill in on each of our character tables. A 1-dimensional representation of G is the same thing as a homomorphism $G \to \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, such a homomorphism can't tell the difference between gh and hg. The following construction measures how far G is from being abelian.

DEFINITION 8.1

Let G be a group and let $g, h \in G$. The **commutator** of g and h is

$$[g, h] := ghg^{-1}h^{-1}.$$

The **commutator subgroup** $[G, G] \subseteq G$ is the subgroup generated by all commutators.

Note that $[g, h] = 1$ if and only if $gh = hg$, and $[g, h]^{-1} = [h, g]$. Hence every element of $[G, G]$ is a finite product of commutators. Be careful: a product of two commutators need not be a commutator itself.

PROBLEM 8.3

Let G be a group.

1. Prove that G is abelian if and only if $[G, G] = \{1\}$.
2. Prove that $k[g, h]k^{-1} = [kgk^{-1}, khk^{-1}]$ for all $g, h, k \in G$. Conclude that $[G, G]$ is a normal subgroup of G.
3. Let $N \subseteq G$ be a normal subgroup. Prove that G/N is abelian if and only if $[G, G] \subseteq N$. In particular, $G^{\text{ab}} := G/[G, G]$ is abelian. We call G^{ab} the **abelianization** of G.
4. Let A be an abelian group and let $\varphi : G \to A$ be a homomorphism. Let $\pi : G \to G^{\text{ab}}$ be the quotient map. Prove that there is a unique homomorphism $\overline{\varphi} : G^{\text{ab}} \to A$ such that $\varphi = \overline{\varphi} \circ \pi$. In other words, every homomorphism from G to an abelian group factors through the abelianization.
5. Prove that $[S_3, S_3] = \langle (123) \rangle$.

8.3 · Pulling Back Representations

If $\varphi : G \to H$ is a homomorphism and $\rho_V : H \to \text{Aut}(V)$ is a representation of H, then $\rho_V \circ \varphi : G \to \text{Aut}(V)$ is a representation of G, called the **pullback** of V along φ. Concretely, $g \in G$ acts on V by $gv := \varphi(g)v$. Pullbacks let us transport representations of a group we understand, such as a quotient of G, back to G itself. The character of $\rho_V \circ \varphi$ is $\chi_V \circ \varphi$.

PROBLEM 8.4

Let $\varphi : G \to H$ be a homomorphism of finite groups.

1. Suppose that φ is surjective and let $f_1, f_2 \in \text{CF}(H)$. Prove that $f_1 \circ \varphi$ and $f_2 \circ \varphi$ are class functions on G and
$$\langle f_1 \circ \varphi, f_2 \circ \varphi \rangle_G = \langle f_1, f_2 \rangle_H.$$
Hint: How many elements of G map to each $h \in H$?

2. Continue to assume φ is surjective. Prove that if V is an irrep of H, then its pullback is an irrep of G, and that pullbacks of non-isomorphic irreps of H are non-isomorphic irreps of G.

3. Prove that two 1-dimensional representations of G are isomorphic if and only if they are equal as homomorphisms $G \to \mathbb{C}^\times$. Then prove that the number of 1-dimensional representations of a finite group G is $|G^{\text{ab}}|$. **Hint:** Problems 6.12 and 7.6.6.

4. Use the previous part to give a quick proof that **1** and **Sgn** are the only 1-dimensional representations of S_3.

8.4 · The Dihedral Group D_4

Let D_4 be the symmetry group of a square, which has order 8. Write r for the counterclockwise rotation by 90° and s for a reflection, so that

$$D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}, \qquad r^4 = s^2 = 1, \quad srs = r^{-1}.$$

Recall that D_4 has exactly 5 conjugacy classes, represented by 1, r^2, r, s, and rs.

PROBLEM 8.5

1. Prove that $[D_4, D_4] = \{1, r^2\}$ and that $D_4^{\text{ab}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Use this to find all 1-dimensional representations of D_4.

2. Prove that D_4 has exactly one irrep which is not 1-dimensional, and that this irrep has dimension 2. Compute the values of this character.
   You may be able to think of a 2-dimensional representation of D_4, but I challenge you to compute the missing character *without* writing down this representation.
   Hint: Column orthogonality.

3. Bring it all home and fill in the character table of D_4. Order the rows so that **1** comes first and the 2-dimensional irrep comes last.

D_4	1	r^2	r	s	rs
1					
χ_2					
χ_3					
χ_4					
χ_5					

8.5 · Tensor Products of Vector Spaces

To compute more character tables, we need more ways of building representations. If V and W are representations, then $V \oplus W$ is a representation whose character is $\chi_V + \chi_W$. Is there a representation

whose character is $\chi_V \chi_W$? Problem 8.2.1 suggests there should be, at least for permutation representations. The construction which does the job is the *tensor product*. We first build the tensor product of vector spaces, and then add in the group action.

DEFINITION 8.2

Let V and W be finite dimensional vector spaces over $\mathbb{C}$ with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively. The **tensor product** $V \otimes W$ is the mn-dimensional vector space with basis given by the symbols

$$v_i \otimes w_j \qquad \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

If $v = \sum_i a_i v_i \in V$ and $w = \sum_j b_j w_j \in W$, we define

$$v \otimes w := \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \, (v_i \otimes w_j) \in V \otimes W.$$

Elements of $V \otimes W$ of the form $v \otimes w$ are called **pure tensors**.

So a typical element of $V \otimes W$ is a linear combination $\sum_{i,j} c_{ij} \, v_i \otimes w_j$. It is helpful to think of it as the $m \times n$ matrix of coefficients (c_{ij}).

PROBLEM 8.6

Let V and W be as in the definition above.

1. Prove that the map $V \times W \to V \otimes W$ given by $(v, w) \mapsto v \otimes w$ is **bilinear**, meaning that for all $v, v' \in V$, $w, w' \in W$, and $c \in \mathbb{C}$,

$$(v + v') \otimes w = v \otimes w + v' \otimes w, \qquad v \otimes (w + w') = v \otimes w + v \otimes w',$$
$$(cv) \otimes w = c(v \otimes w), \qquad v \otimes (cw) = c(v \otimes w).$$

   Is this map linear?

2. Suppose that $v'_1, \ldots, v'_m$ and $w'_1, \ldots, w'_n$ are other bases of V and W. Prove that the pure tensors $v'_i \otimes w'_j$ form a basis of $V \otimes W$. Thus, while we used bases to construct $V \otimes W$, any other choice of bases would have given us the same space and the same pure tensors.

3. (**Universal property**) Let U be a vector space and let $\beta : V \times W \to U$ be bilinear. Prove that there is a unique linear map $\widetilde{\beta} : V \otimes W \to U$ such that $\widetilde{\beta}(v \otimes w) = \beta(v, w)$ for all $v \in V$ and $w \in W$.

The universal property is the best way to think about tensor products: $V \otimes W$ is the space through which every bilinear map out of $V \times W$ factors. It is also the most convenient way to construct linear maps out of $V \otimes W$: to define a linear map on $V \otimes W$, it suffices to give a bilinear map on $V \times W$.

PROBLEM 8.7

Let V, V', W, W', and U be finite dimensional vector spaces.

1. Let $A : V \to V'$ and $B : W \to W'$ be linear maps. Prove that there is a unique linear map $A \otimes B : V \otimes W \to V' \otimes W'$ such that

$$(A \otimes B)(v \otimes w) = Av \otimes Bw$$

for all $v \in V$, $w \in W$.

2. Suppose further that $A' : V' \to V''$ and $B' : W' \to W''$ are linear. Prove that $(A' \otimes B') \circ (A \otimes B) = (A'A) \otimes (B'B)$ and that $I_V \otimes I_W = I_{V \otimes W}$.

3. Now suppose $A : V \to V$ and $B : W \to W$. Prove that

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B).$$

Hint: Write $Av_k = \sum_i a_{ik} v_i$ and $Bw_\ell = \sum_j b_{j\ell} w_j$, and compute the coefficient of $v_k \otimes w_\ell$ in $(A \otimes B)(v_k \otimes w_\ell)$.

4. Prove that there are isomorphisms

$$\begin{aligned} V \otimes W &\cong W \otimes V, & v \otimes w &\mapsto w \otimes v, \\ (U \oplus V) \otimes W &\cong (U \otimes W) \oplus (V \otimes W), & (u, v) \otimes w &\mapsto (u \otimes w, v \otimes w), \\ \mathbb{C} \otimes V &\cong V, & c \otimes v &\mapsto cv. \end{aligned}$$

Hint: Universal property.

8.6 · Tensor Products of Representations

DEFINITION 8.3

Let V and W be representations of G. The **tensor product representation** is $V \otimes W$ with G acting by

$$\rho_{V \otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

On pure tensors, $g(v \otimes w) = gv \otimes gw$.

PROBLEM 8.8

Let V and W be finite dimensional representations of a finite group G.

1. Prove that $\rho_{V \otimes W}$ is a representation of G and that

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

2. Let U be another representation of G. Prove that the three isomorphisms in Problem 8.7.4 are G-linear, where $\mathbb{C}$ is given the trivial action. Thus $V \otimes W \cong W \otimes V$, $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$, and $\mathbf{1} \otimes V \cong V$ as representations.

3. Let L be a 1-dimensional representation of G and let W be an irrep of G. Prove that $|\chi_L(g)| = 1$ for all $g \in G$, and use this to prove that $L \otimes W$ is irreducible.

4. Let X and Y be finite G-sets. Prove that $\mathbb{C}(X \times Y) \cong \mathbb{C}X \otimes \mathbb{C}Y$ as representations of G.

5. The previous part says that the operation $X \mapsto \mathbb{C}X$ sends sums of G-sets to direct sums and products of G-sets to tensor products. Decompose the following S_3-representations into a sum of irreps,

$$\mathbf{Sgn} \otimes \mathbf{Sgn}, \qquad \mathbf{Sgn} \otimes \mathbf{Std}, \qquad \mathbf{Std} \otimes \mathbf{Std}.$$

Hint: There's a slick way to to this using your multiplication table for $B(S_3)$ from Problem 4.4.2 together with Problem 8.1.2.

8.7 · The Character Table of S_4

We now have all the tools we need to compute the character table of S_4. Recall that two permutations in S_n are conjugate if and only if they have the same cycle type.

PROBLEM 8.9

1. Show that S_4 has five conjugacy classes, represented by ε, (12), (12)(34), (123), and (1234), and compute the size of each class.

2. Compute the characters of $\mathbf{1}$, $\mathbf{Sgn}$, $\mathbf{Std}_4$, and $\mathbf{Std}_4 \otimes \mathbf{Sgn}$, and prove that these are four pairwise non-isomorphic irreps of S_4. Here $\mathbf{Sgn}$ is the 1-dimensional representation $\sigma z := \operatorname{sgn}(\sigma) z$.

3. Prove that S_4 has exactly one irrep W which is not isomorphic to one of the irreps from part (2), that $\dim W = 2$, and compute χ_W.
Hint: Column orthogonality, as in Problem 8.5.2.
Challenge: Can you see how to construct this 2-dimensional representation of S_4?

4. Fill in the character table of S_4.

S_4	ε	(12)	(12)(34)	(123)	(1234)
$\mathbf{1}$					
$\mathbf{Sgn}$					
$\mathbf{Std}_4$					
$\mathbf{Std}_4 \otimes \mathbf{Sgn}$					
W					

5. Let $\binom{[4]}{2}$ be the set of 2-element subsets of $\{1, 2, 3, 4\}$, with the action of S_4 by relabeling. Decompose each of the following representations of S_4 into irreps:

$$\mathbb{C}\binom{[4]}{2} \qquad \text{and} \qquad \mathbf{Std}_4 \otimes \mathbf{Std}_4.$$